First-Order Logic

Part One

Recap from Last Time

Recap So Far

- A ***propositional variable*** is a variable that is either true or false.
- The ***propositional connectives*** are as follows:
  - Negation: $\neg p$
  - Conjunction: $p \land q$
  - Disjunction: $p \lor q$
  - Implication: $p \to q$
  - Biconditional: $p \leftrightarrow q$
  - True: $\top$
  - False: $\bot$

Take out a sheet of paper!

What's the truth table for the $\rightarrow$ connective?

What's the negation of $p \rightarrow q$?

New Stuff!

First-Order Logic

What is First-Order Logic?

- ***First-order logic*** is a logical system for reasoning about properties of objects.
- Augments the logical connectives from propositional logic with
  - ***predicates*** that describe properties of objects,
  - ***functions*** that map objects to one another, and
  - ***quantifiers*** that allow us to reason about multiple objects.

Some Examples

$$Likes(You, Eggs) \wedge Likes(You, Tomato) \rightarrow Likes(You, Shakshuka)$$

$$Likes(You, Eggs) \land Likes(You, Tomato) \rightarrow Likes(You, Shakshuka)$$

$$Learns(You, History) \lor ForeverRepeats(You, History)$$

$$In(MyHeart, Havana) \land TookBackTo(Him, Me, EastAtlanta)$$

$$Likes(You, Eggs) \land Likes(You, Tomato) \rightarrow Likes(You, Shakshuka)$$

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$$In(MyHeart, Havana) \land TookBackTo(Him, Me, EastAtlanta)$$

These blue terms are called ***constant symbols***. Unlike propositional variables, they refer to *objects*, not *propositions*.

$$Likes(You, Eggs) \land Likes(You, Tomato) \rightarrow Likes(You, Shakshuka)$$

$$Learns(You, History) \lor ForeverRepeats(You, History)$$

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$$Learns(You, History) \lor ForeverRepeats(You, History)$$

$$In(MyHeart, Havana) \land TookBackTo(Him, Me, EastAtlanta)$$

The red things that look like function calls are called ***predicates***. Predicates take objects as arguments and evaluate to true or false.

$$Likes(You, Eggs) \land Likes(You, Tomato) \rightarrow Likes(You, Shakshuka)$$

$$Learns(You, History) \lor ForeverRepeats(You, History)$$

$$In(MyHeart, Havana) \land TookBackTo(Him, Me, EastAtlanta)$$

$$Likes(You, Eggs) \land Likes(You, Tomato) \rightarrow Likes(You, Shakshuka)$$

$$Learns(You, History) \lor ForeverRepeats(You, History)$$

$$In(MyHeart, Havana) \land TookBackTo(Him, Me, EastAtlanta)$$

What remains are traditional propositional connectives. Because each predicate evaluates to true or false, we can connect the truth values of predicates using normal propositional connectives.

Reasoning about Objects

- To reason about objects, first-order logic uses ***predicates***.
- Examples:

$$Cute(Quokka)$$

$$Cool(CS103\ students)$$

- Applying a predicate to arguments produces a proposition, which is either true or false.
- Typically, when you're working in FOL, you'll have a list of predicates, what they stand for, and how many arguments they take. It'll be given separately from the formulas you write.

First-Order Sentences

- Sentences in first-order logic can be constructed from predicates applied to objects:

$$Cute(a) \to Dikdik(a) \lor Kitty(a) \lor Puppy(a)$$

$$Succeeds(You) \leftrightarrow Practices(You)$$

$$x < 8 \to x < 137$$

The less-than sign is just another predicate. Binary predicates are sometimes written in ***infix notation*** this way.

Numbers are not “built in” to first-order logic. They’re constant symbols just like “You” and “a” above.

Equality

- First-order logic is equipped with a special predicate **=** that says whether two objects are equal to one another.
- Equality is a part of first-order logic, just as → and ¬ are.
- Examples:

$$MilesMorales = SpiderMan$$

$$MorningStar = EveningStar$$

- Equality can only be applied to ***objects***; to state that two ***propositions*** are equal, use ↔.

Let's see some more examples.

$$FavoriteMovieOf(You) \neq FavoriteMovieOf(Date) \wedge$$
$$StarOf(FavoriteMovieOf(You)) = StarOf(FavoriteMovieOf(Date))$$

$$FavoriteMovieOf(You) \neq FavoriteMovieOf(Date) \land$$
$$StarOf(FavoriteMovieOf(You)) = StarOf(FavoriteMovieOf(Date))$$

$$FavoriteMovieOf(You) \neq FavoriteMovieOf(Date) \wedge StarOf(FavoriteMovieOf(You)) = StarOf(FavoriteMovieOf(Date))$$

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$$FavoriteMovieOf(You) \neq FavoriteMovieOf(Date) \land StarOf(FavoriteMovieOf(You)) = StarOf(FavoriteMovieOf(Date))$$

These purple terms are ***functions***. Functions take objects as input and produce objects as output.

$$FavoriteMovieOf(You) \neq FavoriteMovieOf(Date) \land$$
$$StarOf(FavoriteMovieOf(You)) = StarOf(FavoriteMovieOf(Date))$$

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$$
\mathit{FavoriteMovieOf}(\mathit{You}) \neq \mathit{FavoriteMovieOf}(\mathit{Date}) \land
\mathit{StarOf}(\mathit{FavoriteMovieOf}(\mathit{You})) = \mathit{StarOf}(\mathit{FavoriteMovieOf}(\mathit{Date}))
$$

Functions

- First-order logic allows ***functions*** that return objects associated with other objects.
- Examples:

$$ColorOf(Money)$$

$$MedianOf(x, y, z)$$

$$x + y$$

- As with predicates, functions can take in any number of arguments, but always return a single value.
- Functions evaluate to ***objects***, not ***propositions***.

Objects and Predicates

- When working in first-order logic, be careful to keep objects (actual things) and propositions (true or false) separate.
- You cannot apply connectives to objects:

⚠ $Venus \to TheSun$ ⚠

- You cannot apply functions to propositions:

⚠ $StarOf(IsRed(Sun) \land IsGreen(Mars))$ ⚠

- Ever get confused? *Just ask!*

The Type-Checking Table

	… operate on …	… and produce
Connectives (↔, ∧, etc.) …	propositions	a proposition
Predicates (=, etc.) …	objects	a proposition
Functions …	objects	an object

One last (and major) change

Some spider is radioactive.

Some spider is radioactive.

$$\exists s.\ (Spider(s) \land Radioactive(s))$$

Some spider is radioactive.

$$\exists s.\ (Spider(s) \land Radioactive(s))$$

∃ is the ***existential quantifier*** and says "for some choice of s, the following is true."

The Existential Quantifier

- A statement of the form

$$\exists x.\ \textit{some-formula}$$

  is true if there exists a choice of x where ***some-formula*** is true when that x is plugged into it.

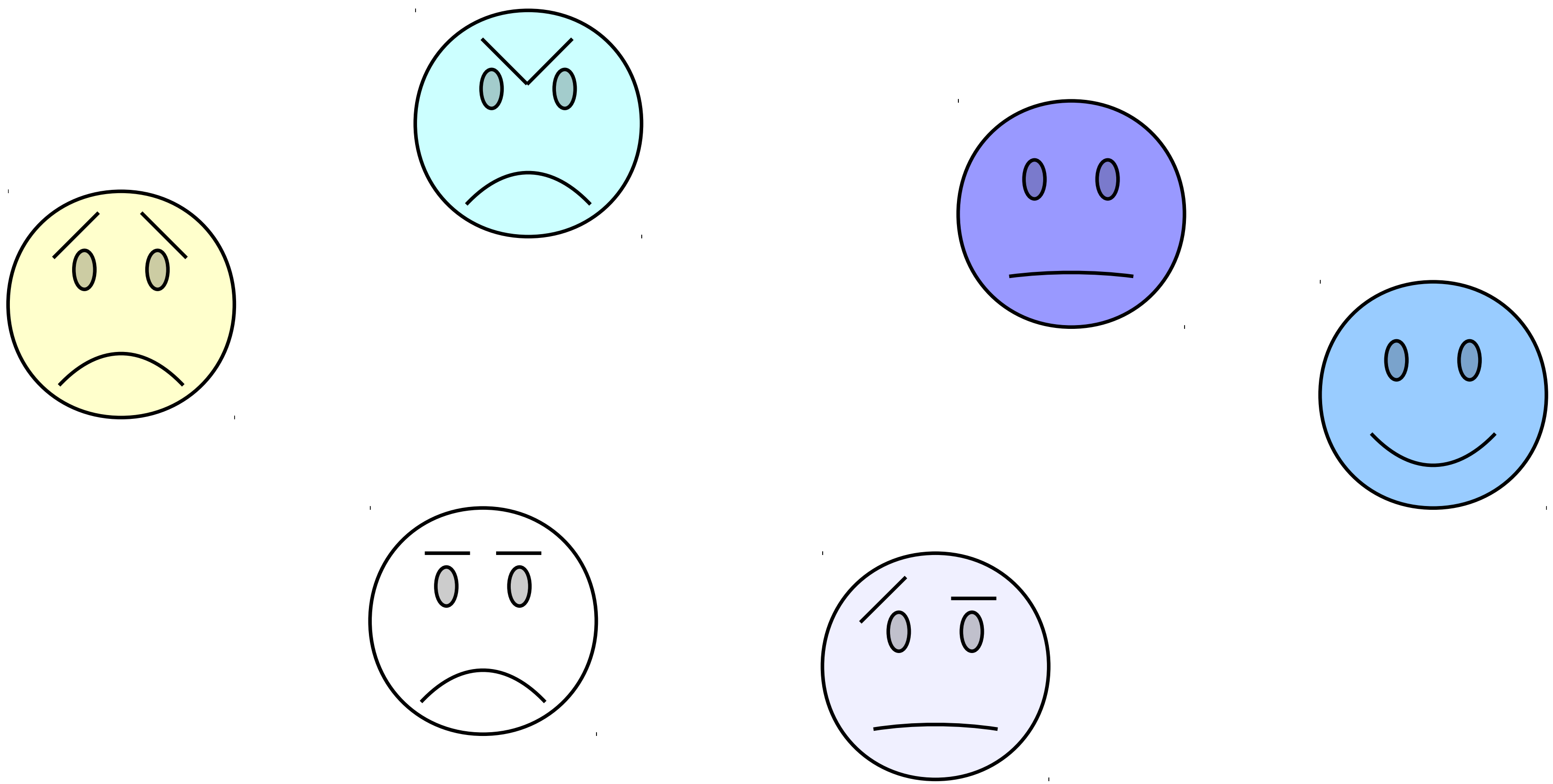
- Examples:

$$\exists x.\ (Even(x) \land Prime(x))$$

$$\exists x.\ (TallerThan(x, me) \land LighterThan(x, me))$$

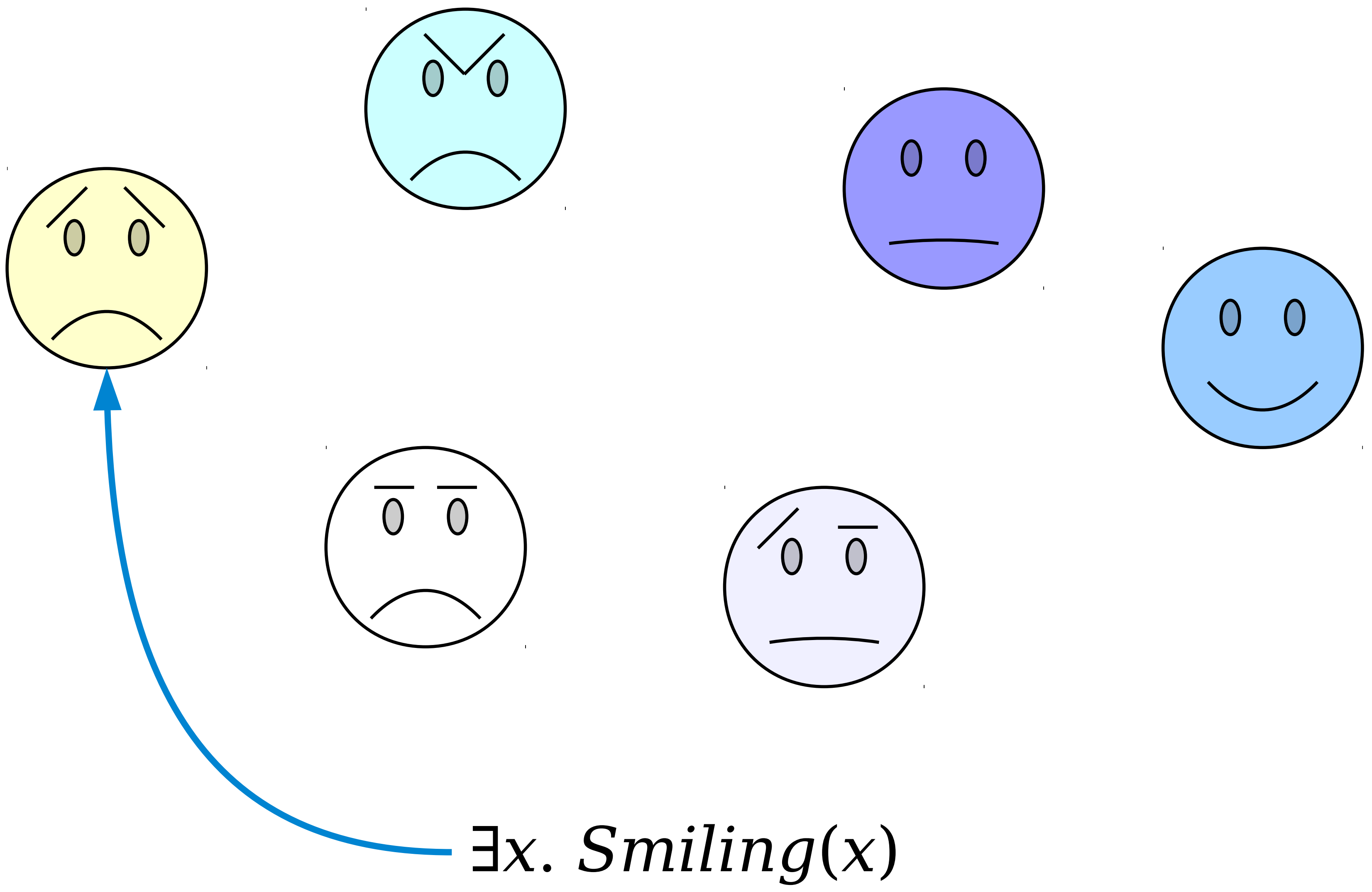
$$(\exists w.\ Will(w)) \rightarrow (\exists x.\ Way(x))$$

The Existential Quantifier

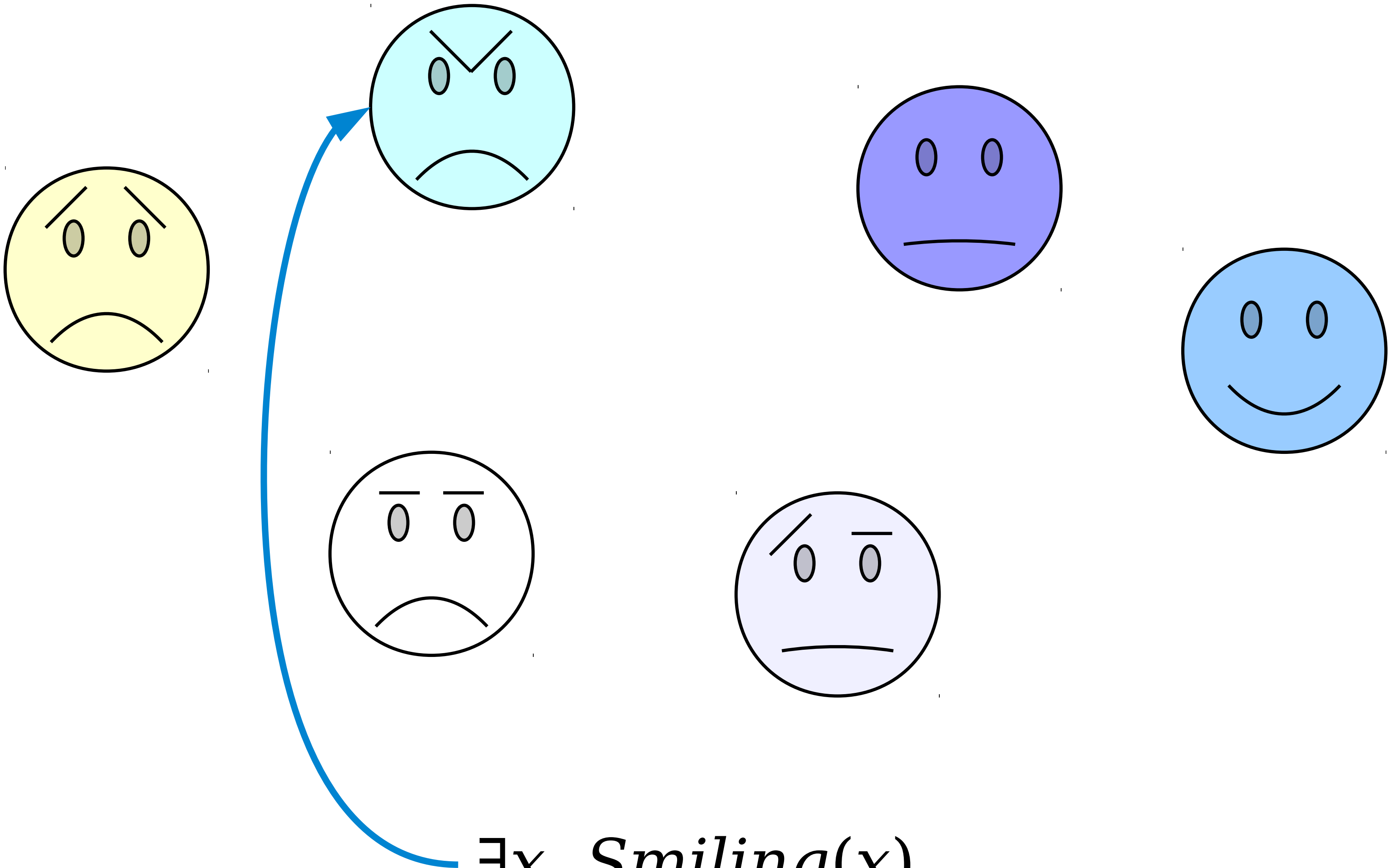


$$\exists x.\ Smiling(x)$$

The Existential Quantifier



The Existential Quantifier



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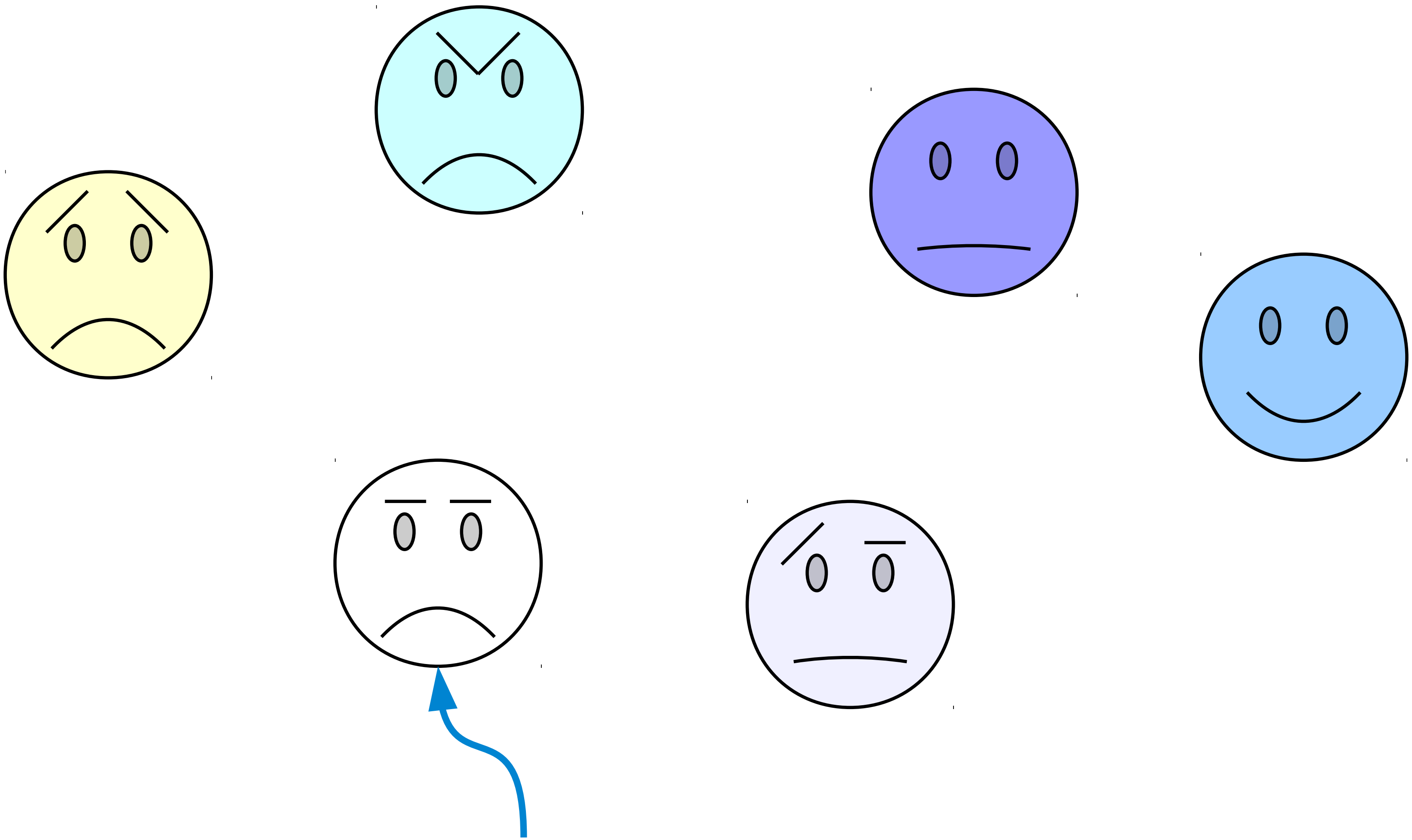

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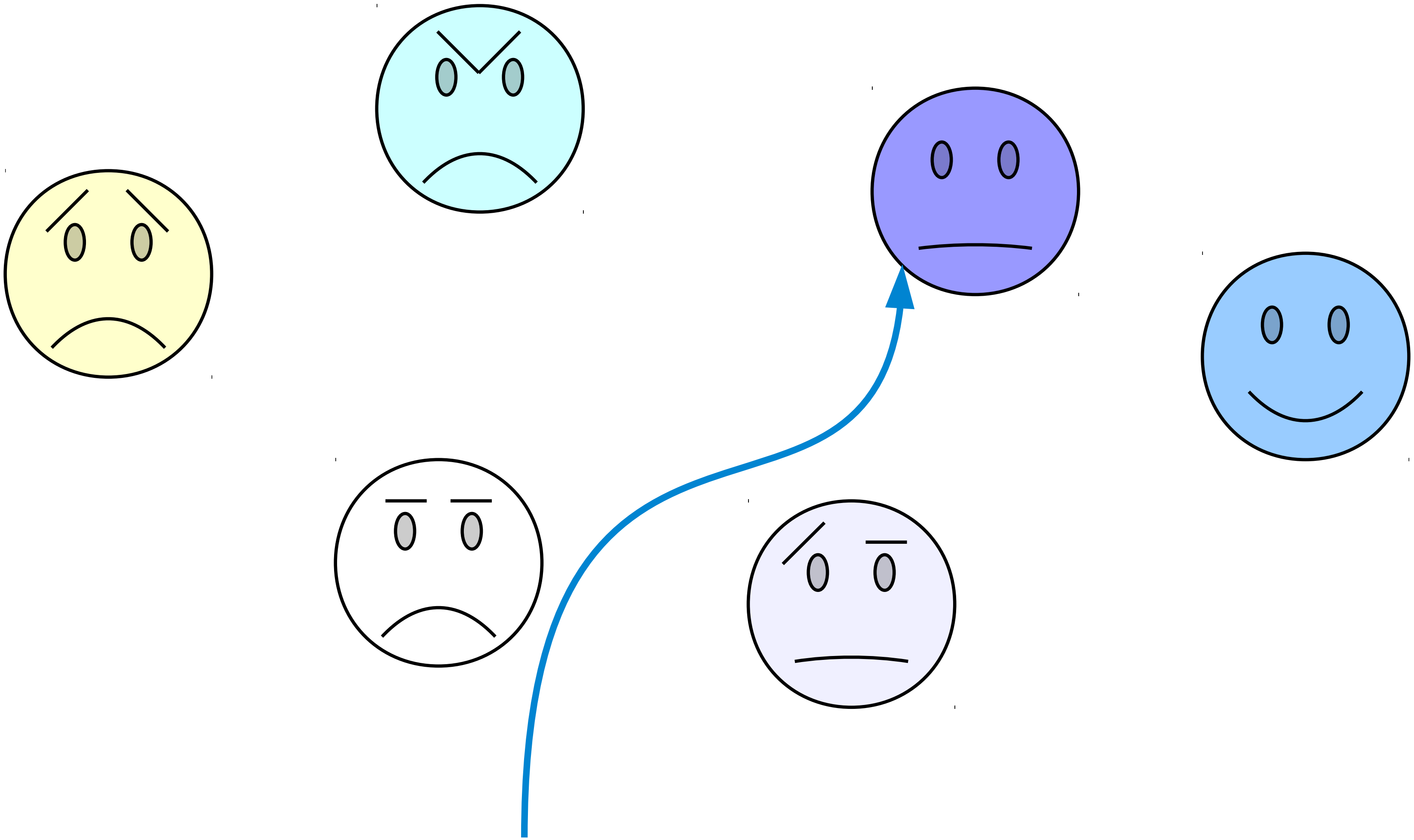

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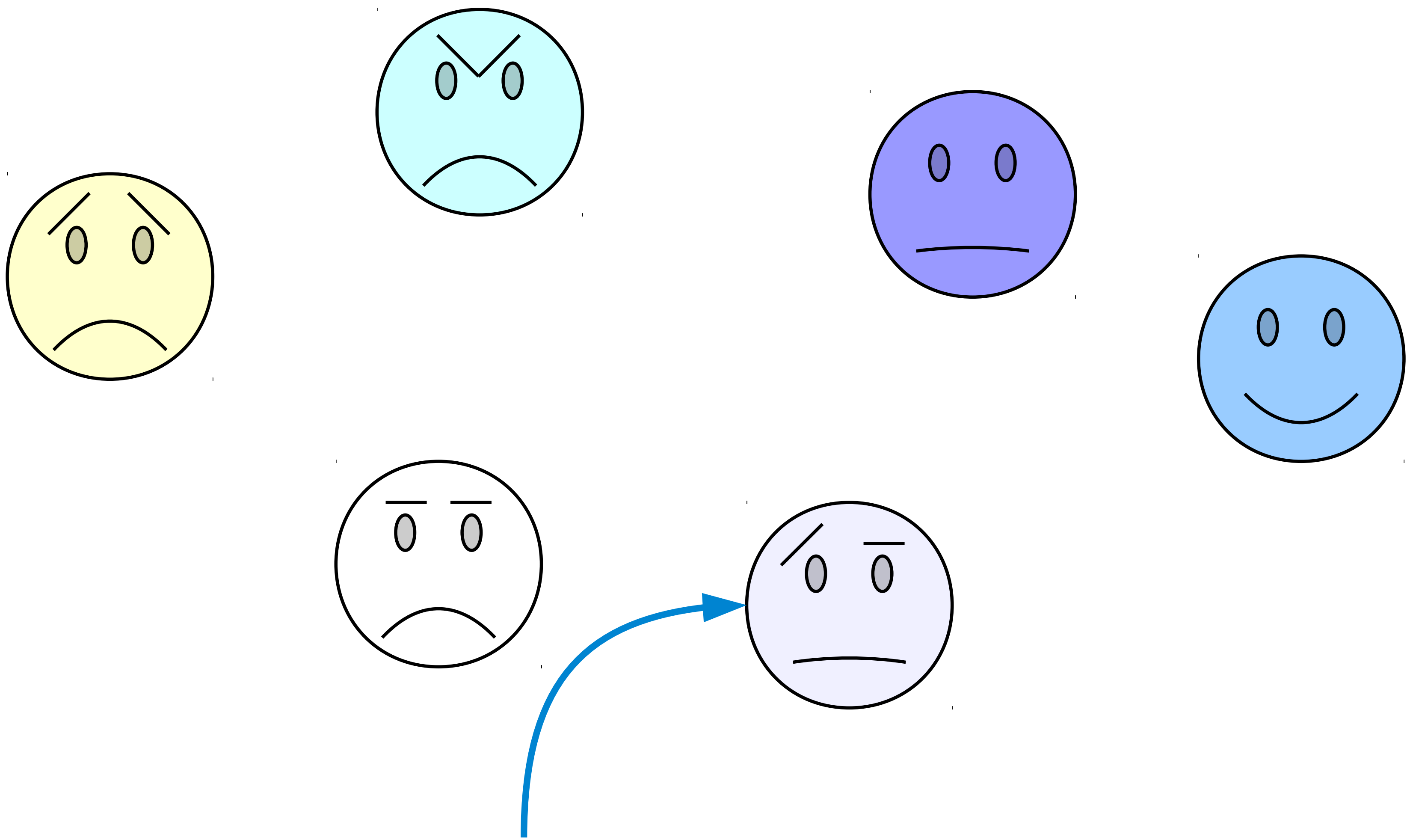

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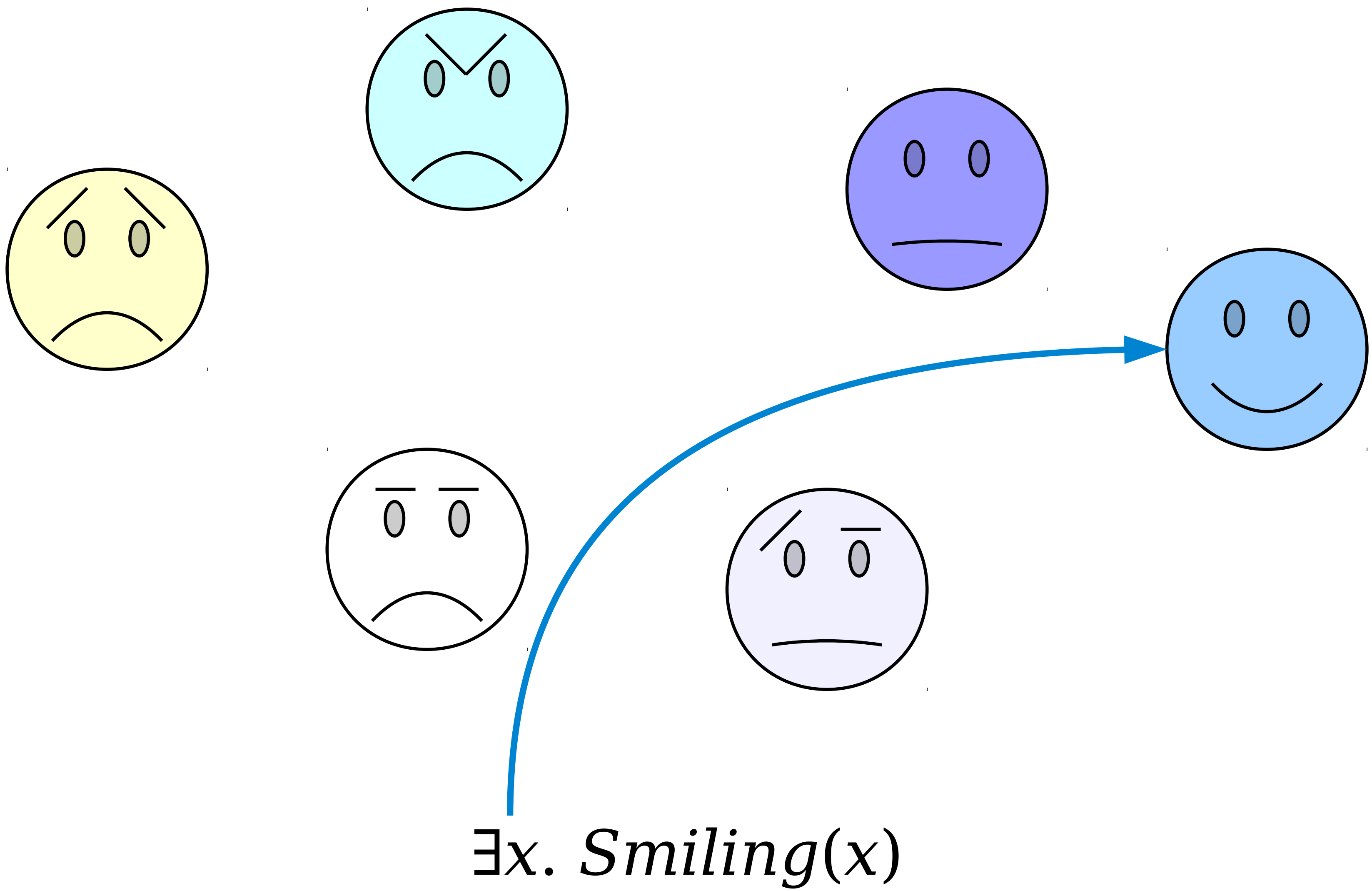

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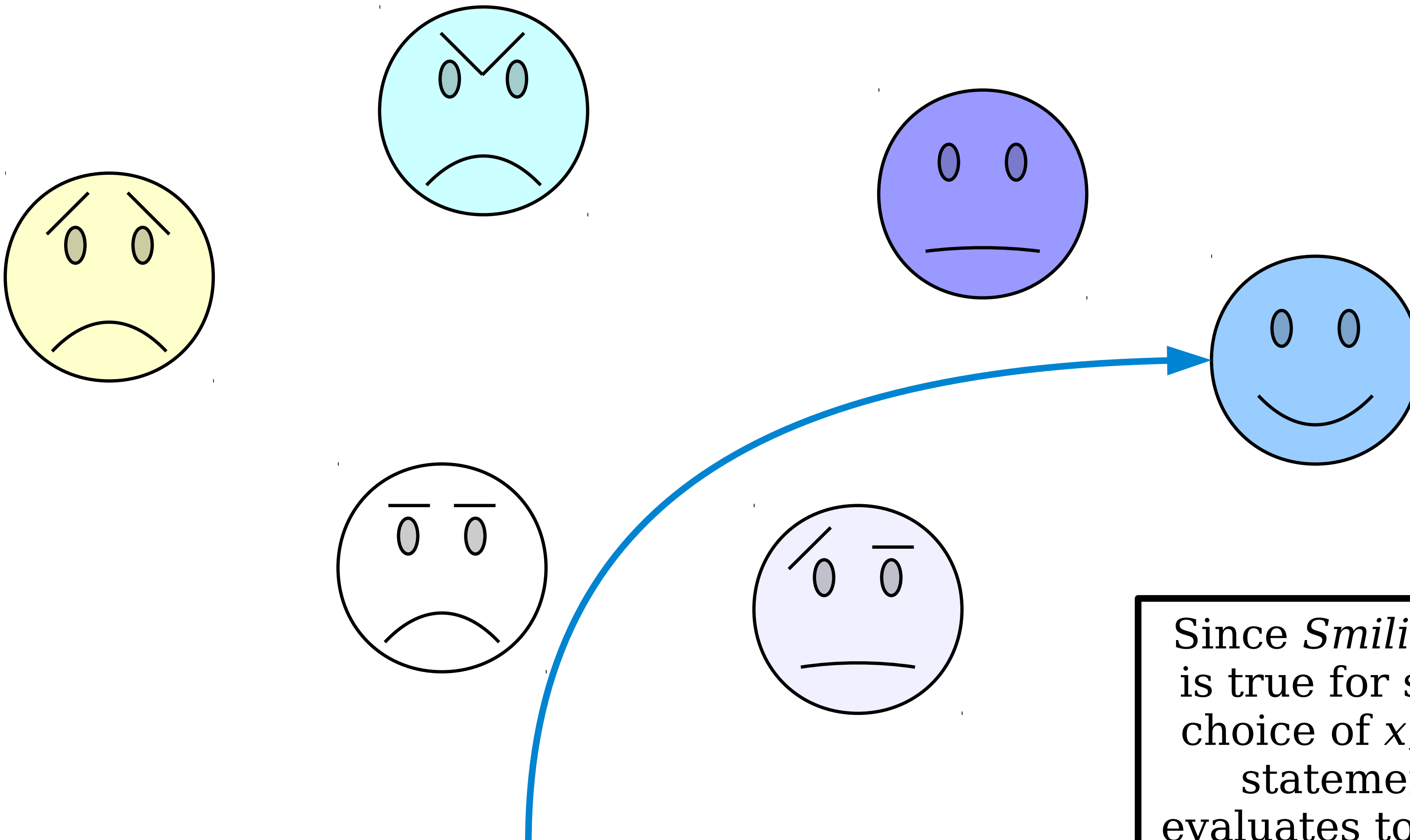
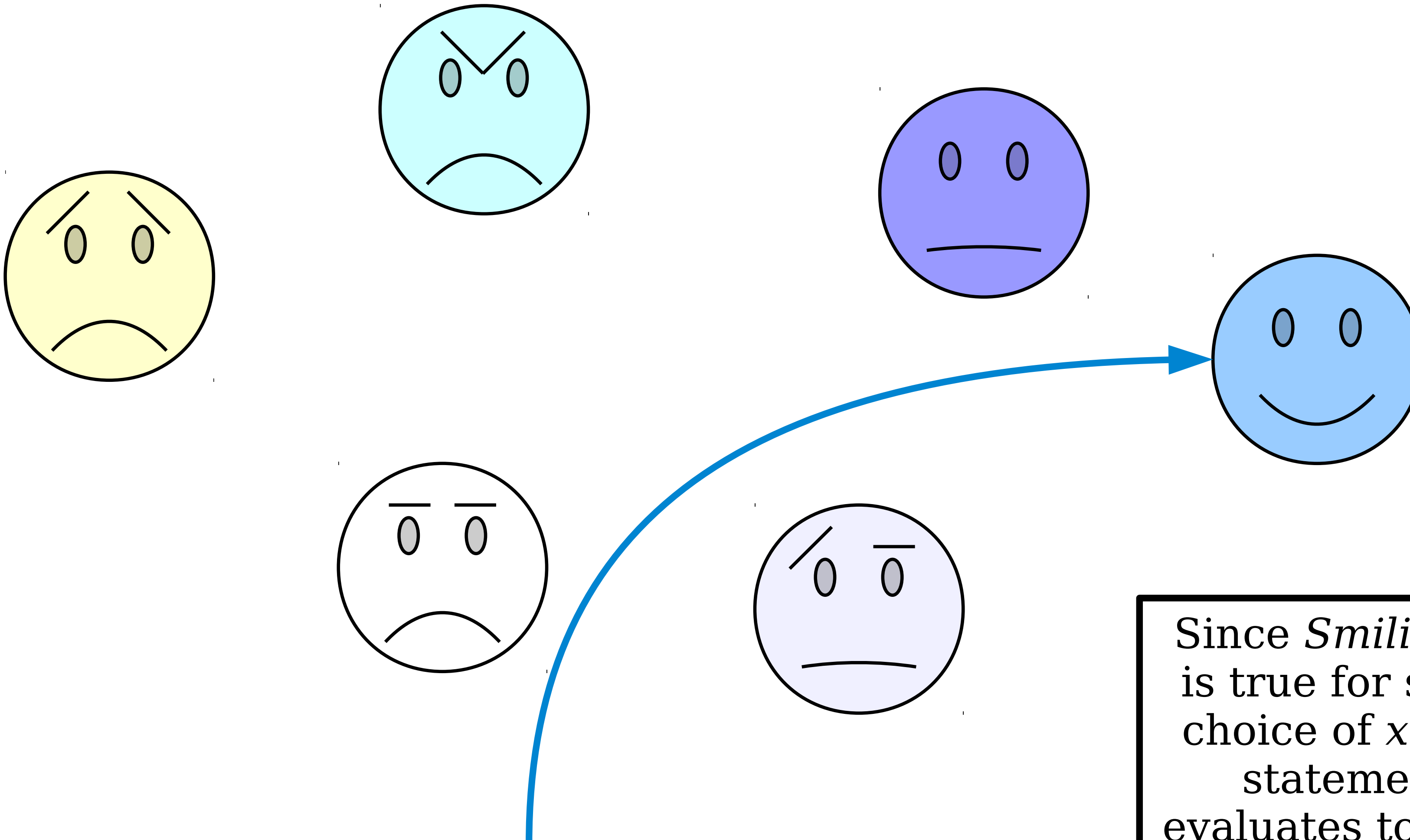


$\exists x.\ Smiling(x)$

Since *Smiling*(x) is true for some choice of x, this statement evaluates to true.

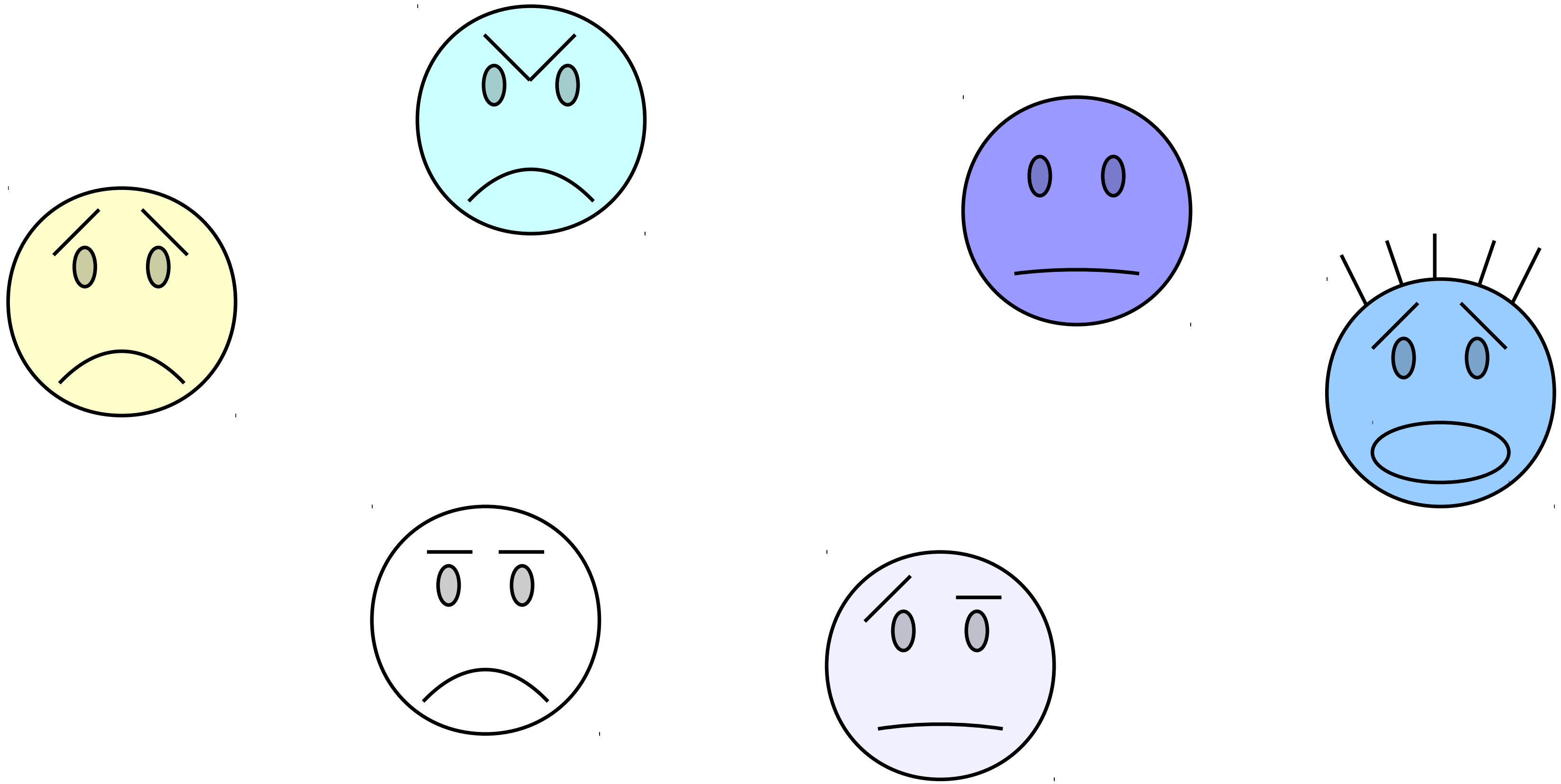
The Existential Quantifier



$\exists x.\ Smiling(x)$

Since $Smiling(x)$ is true for some choice of x, this statement evaluates to true.

The Existential Quantifier



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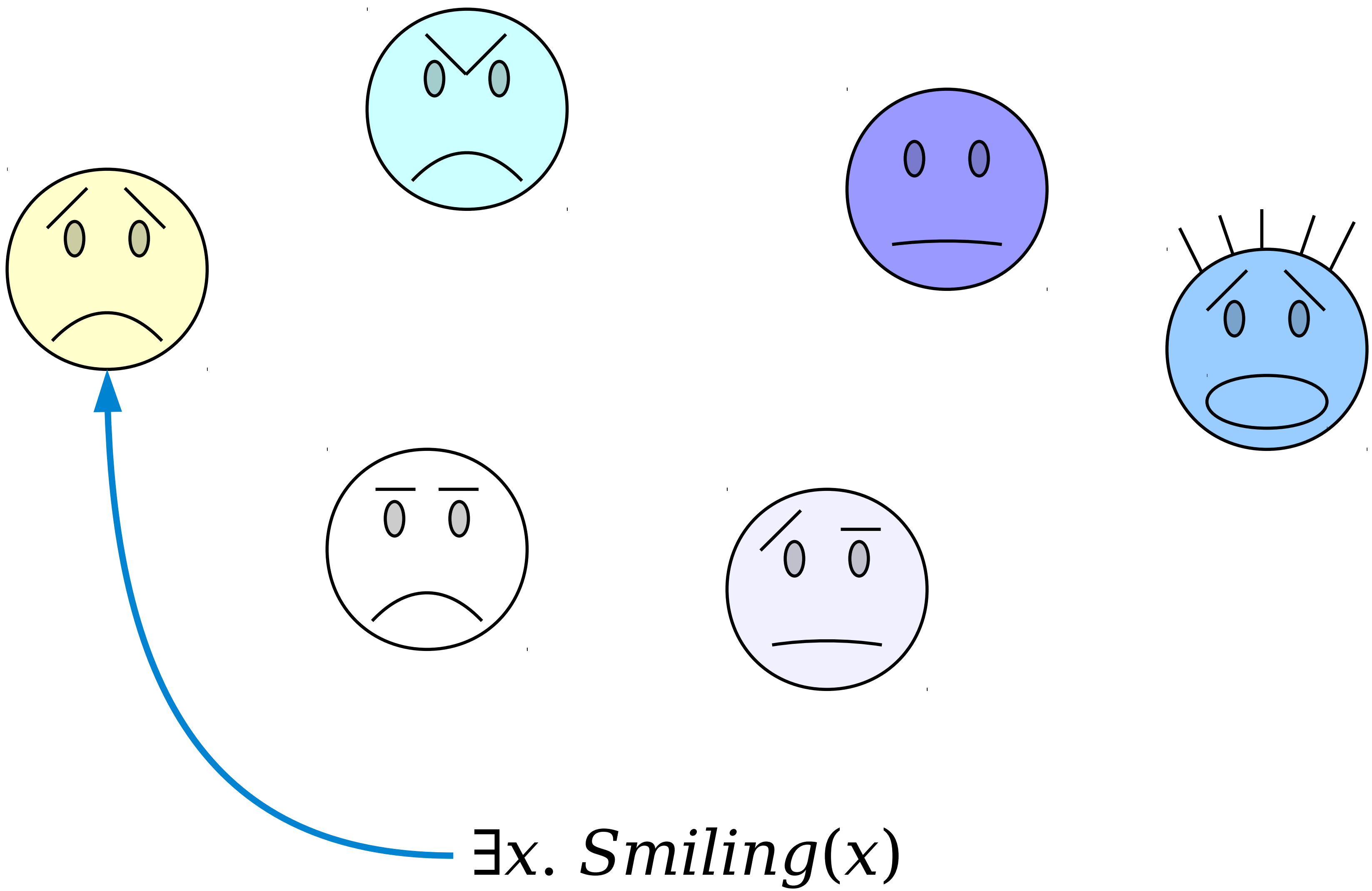

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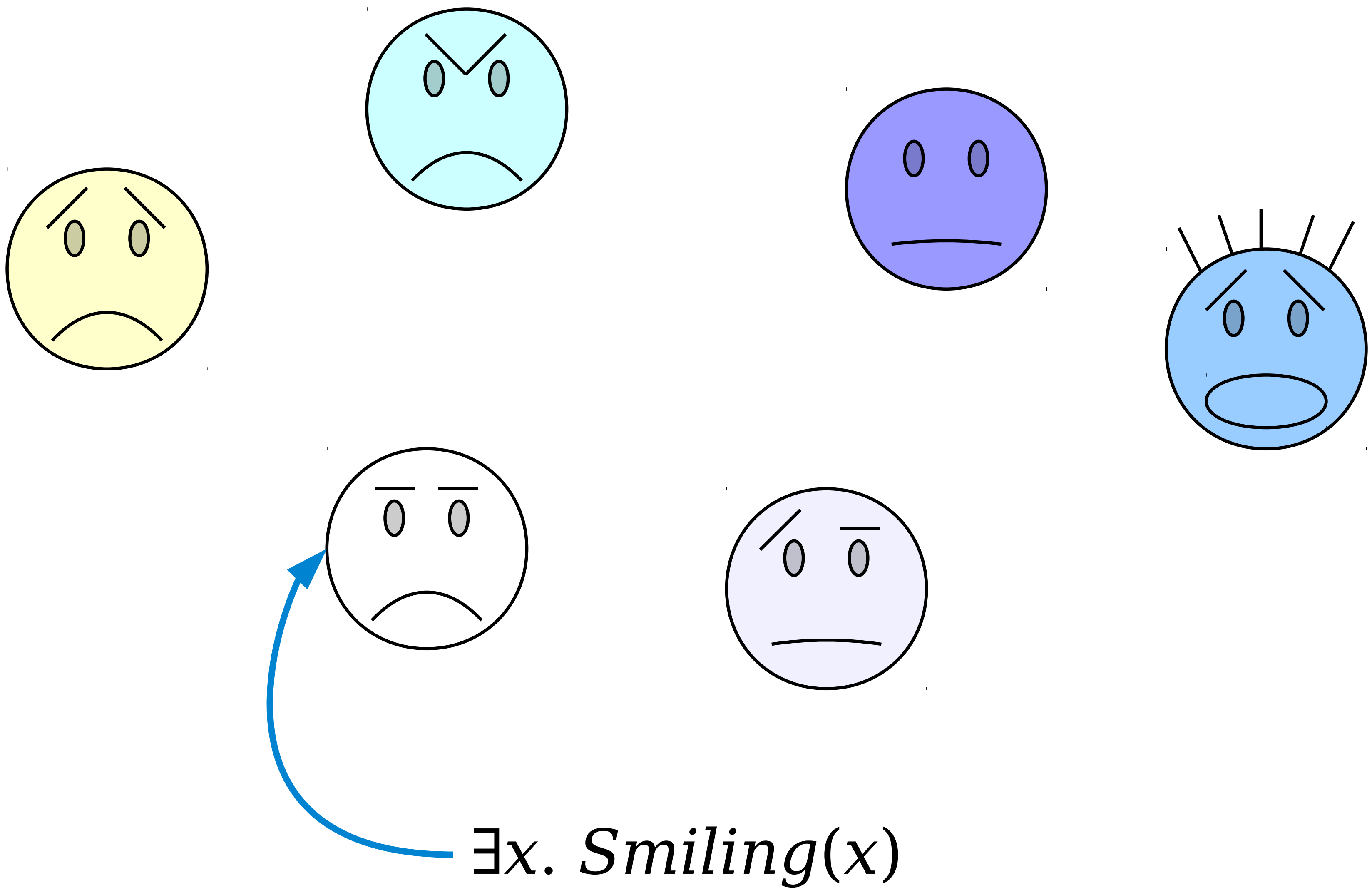

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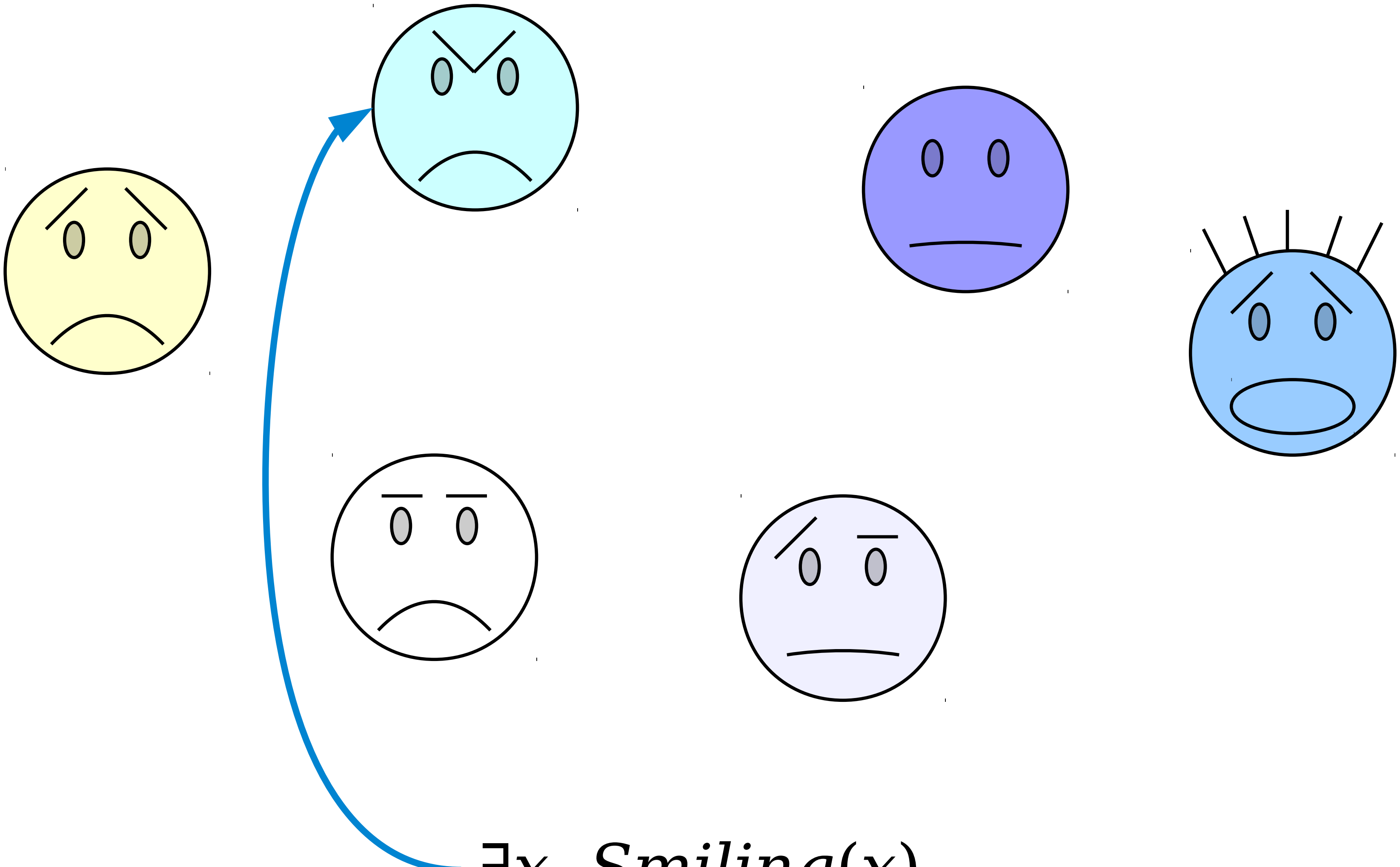

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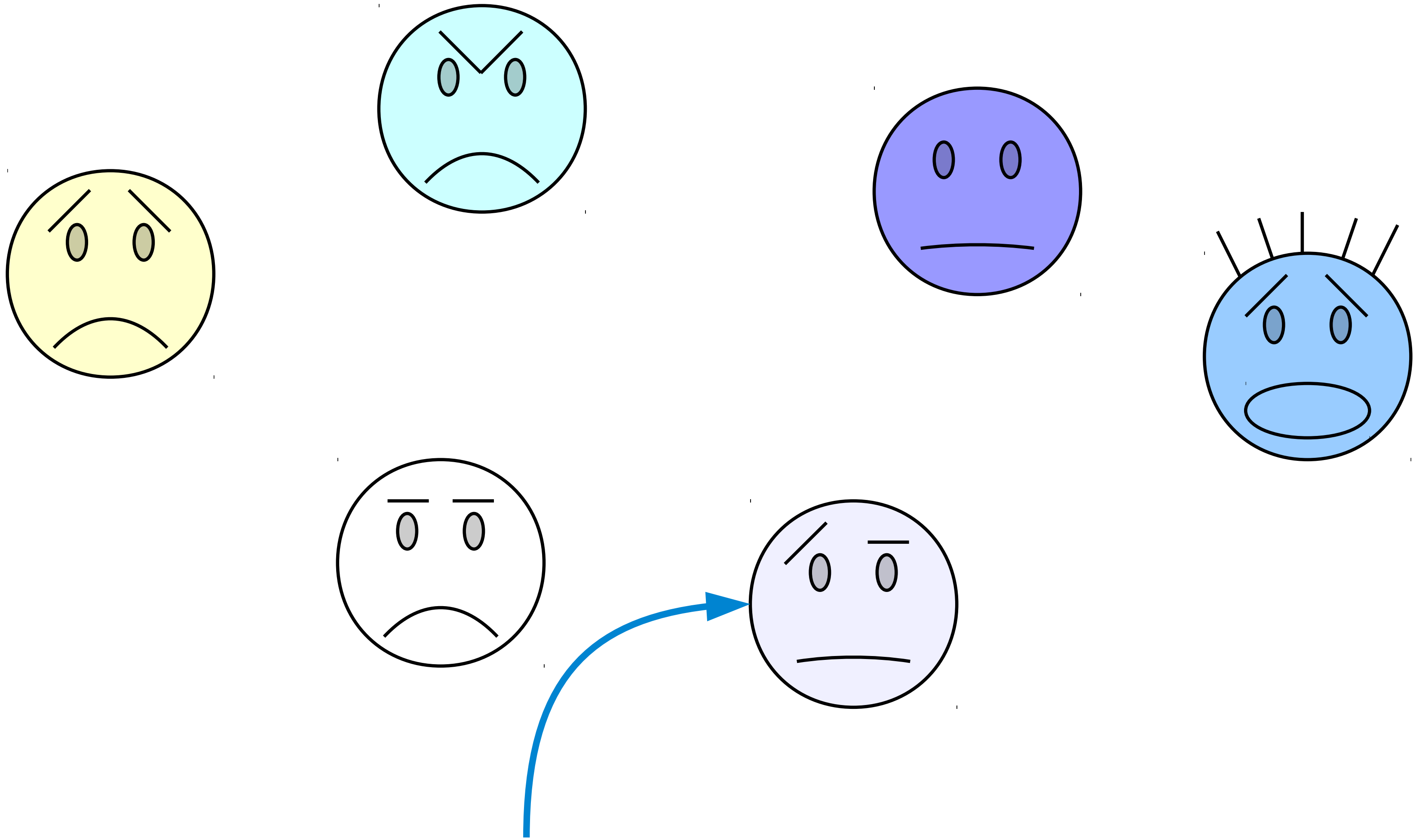

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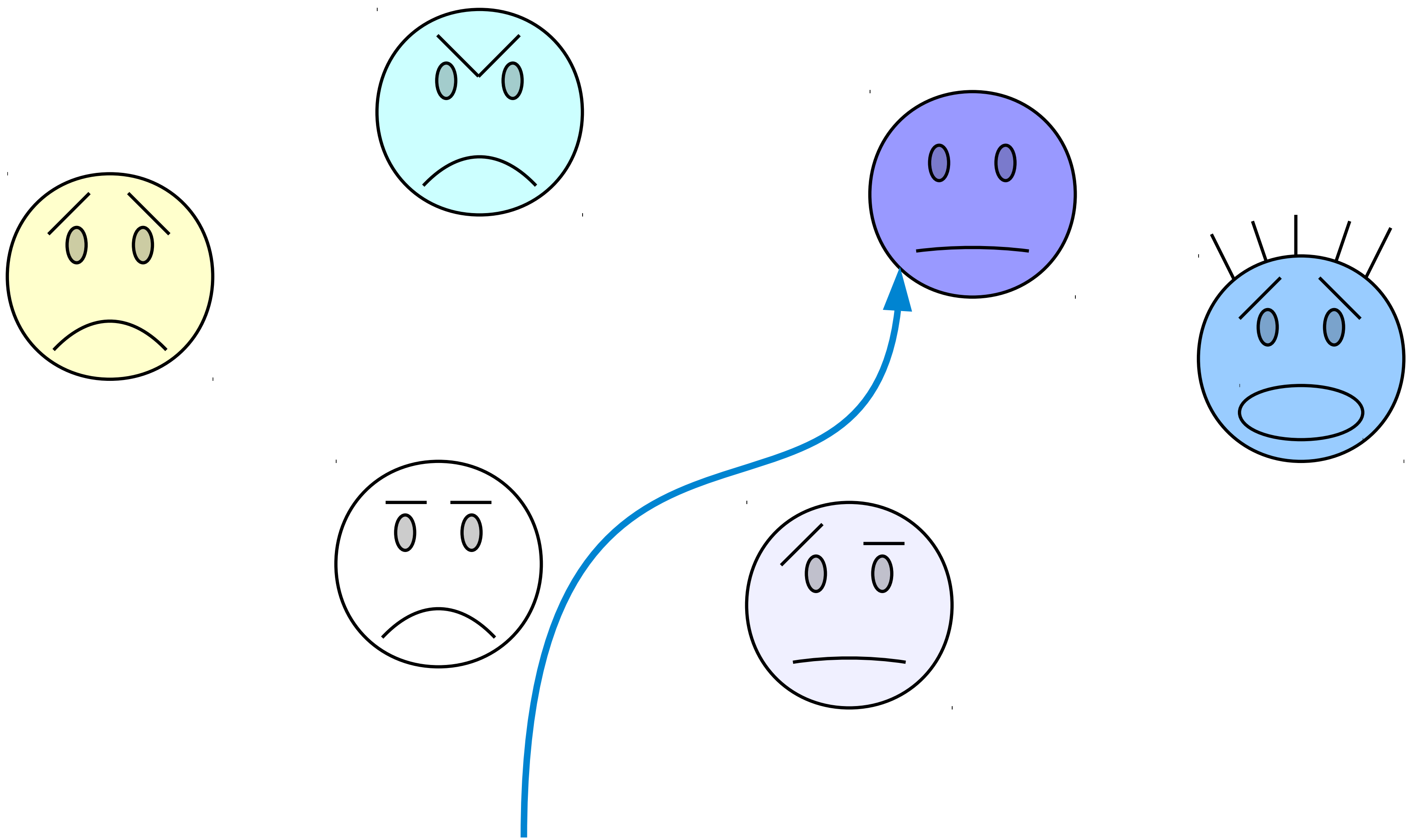
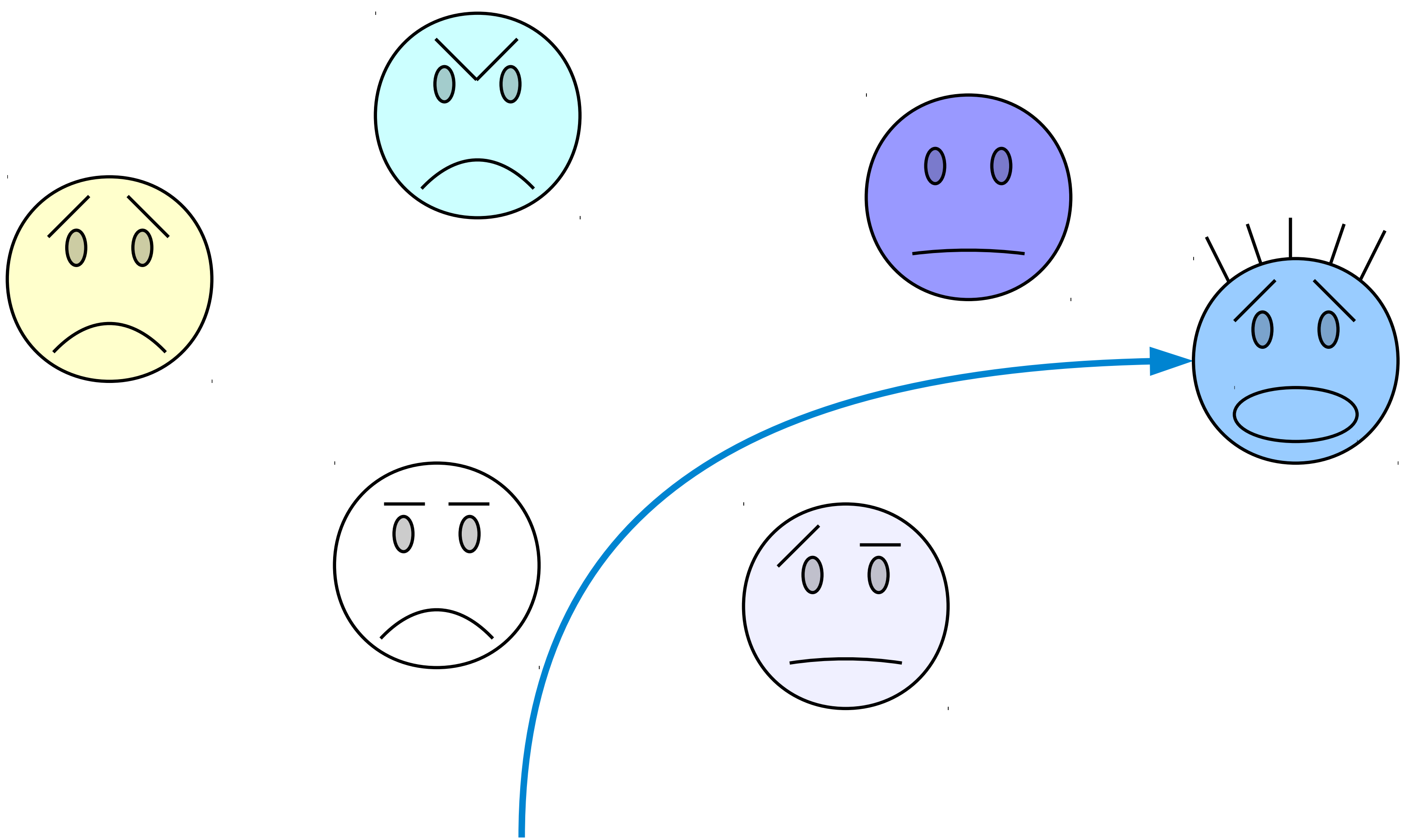


$\exists x.\ \mathit{Smiling}(x)$

The Existential Quantifier



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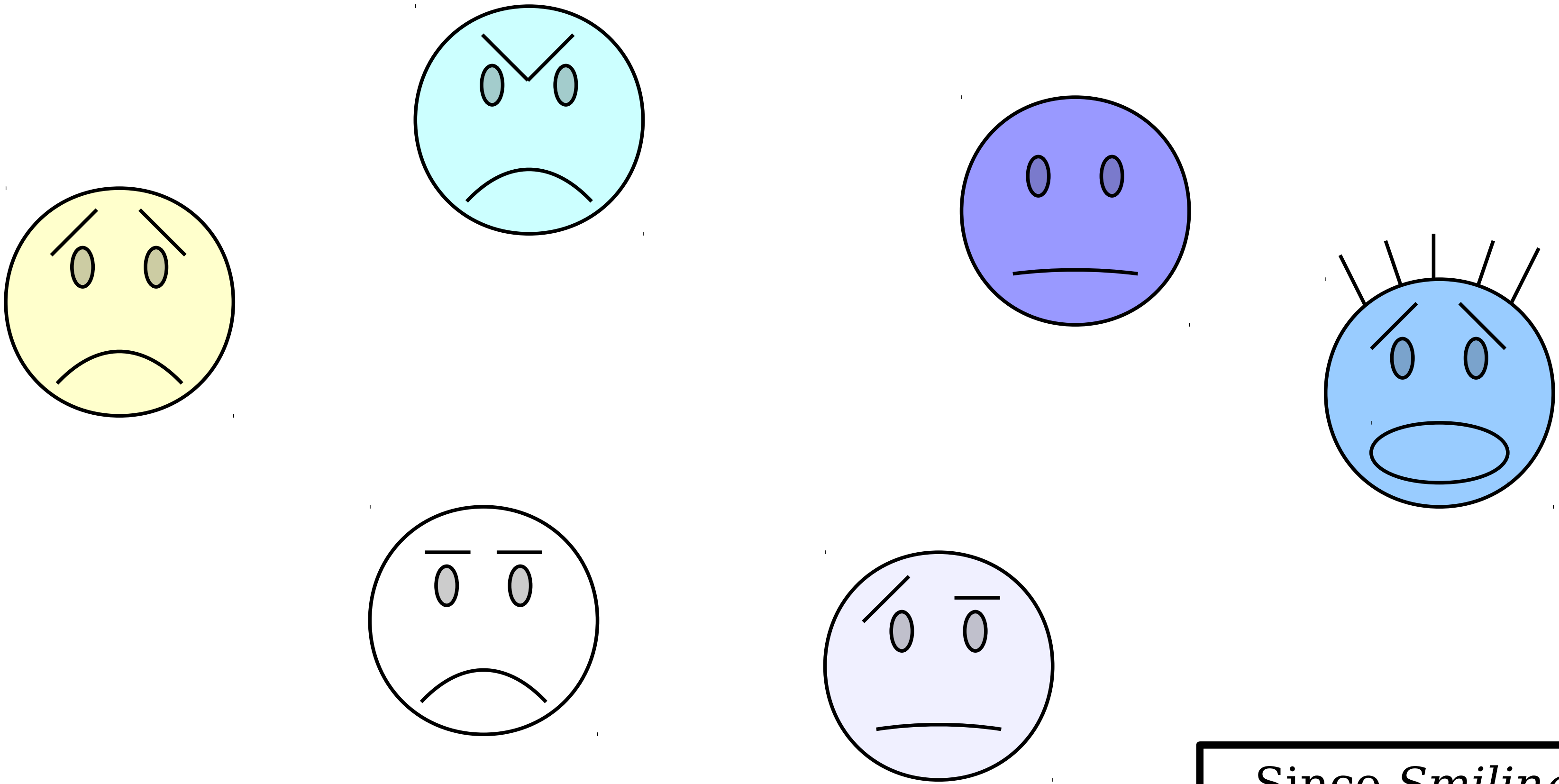
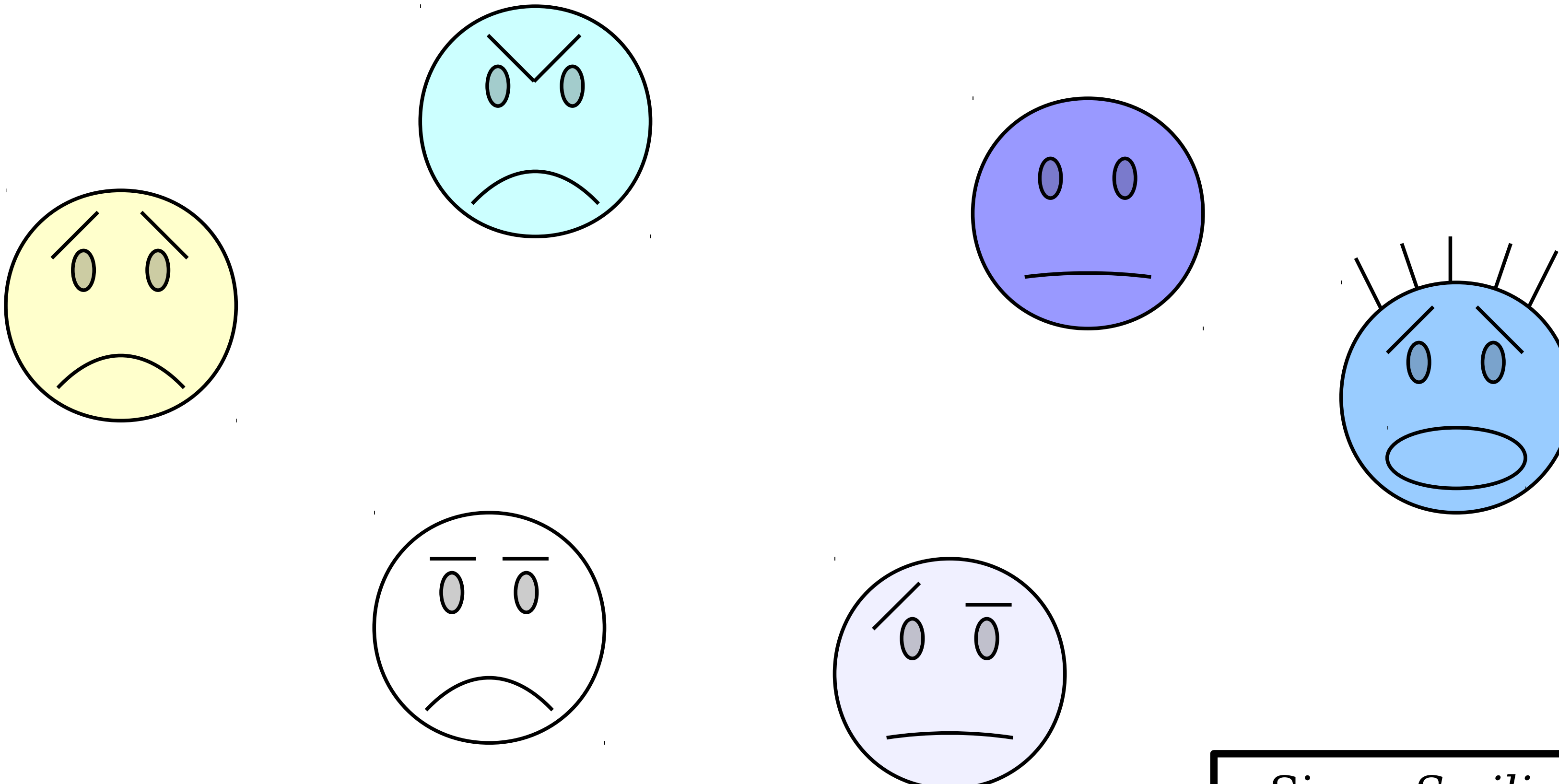


$$\exists x.\ Smiling(x)$$

Since $Smiling(x)$ is not true for any choice of x, this statement evaluates to false.

The Existential Quantifier

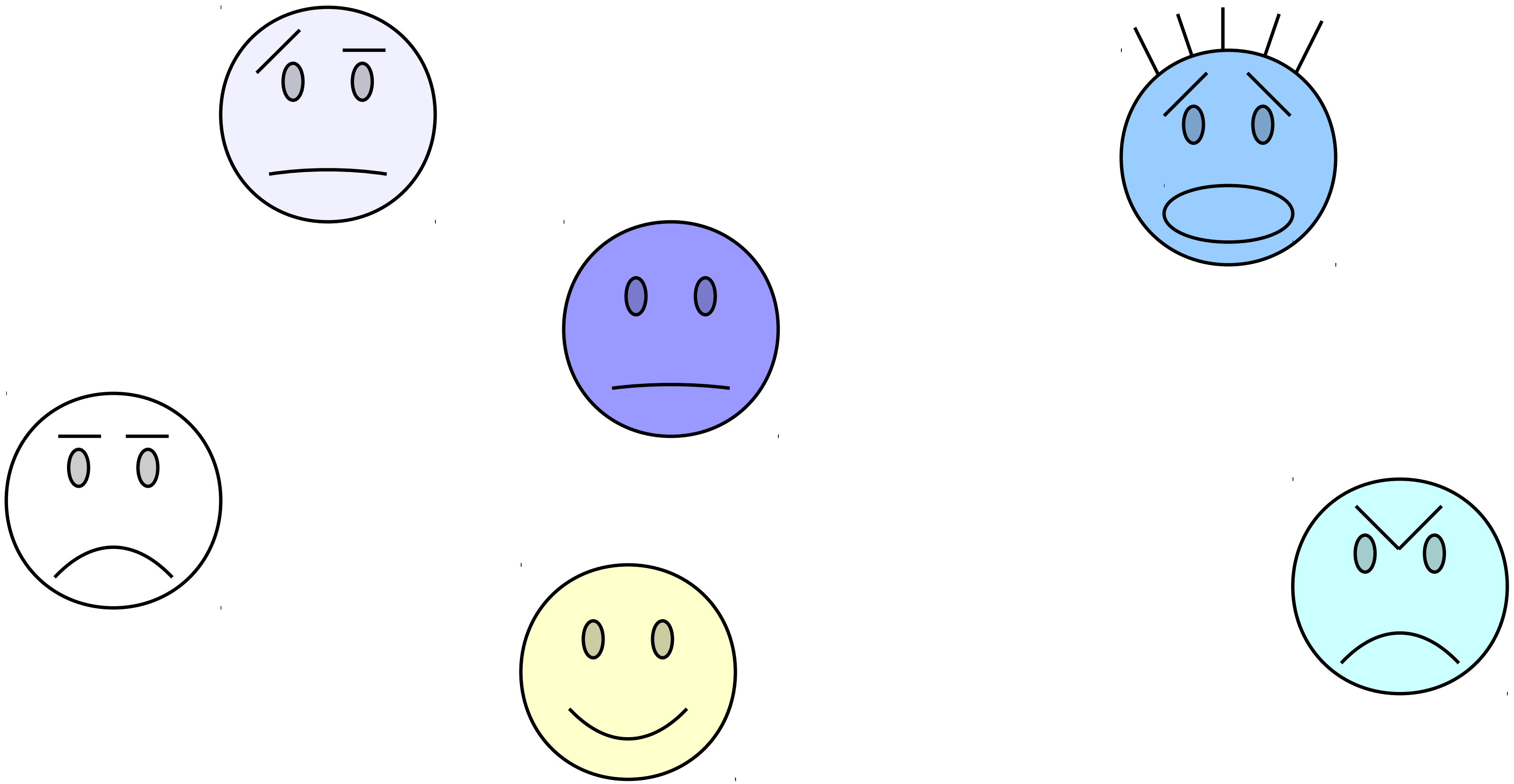


~~$\exists x.\ Smiling(x)$~~

Since $Smiling(x)$ is not true for any choice of x, this statement evaluates to false.

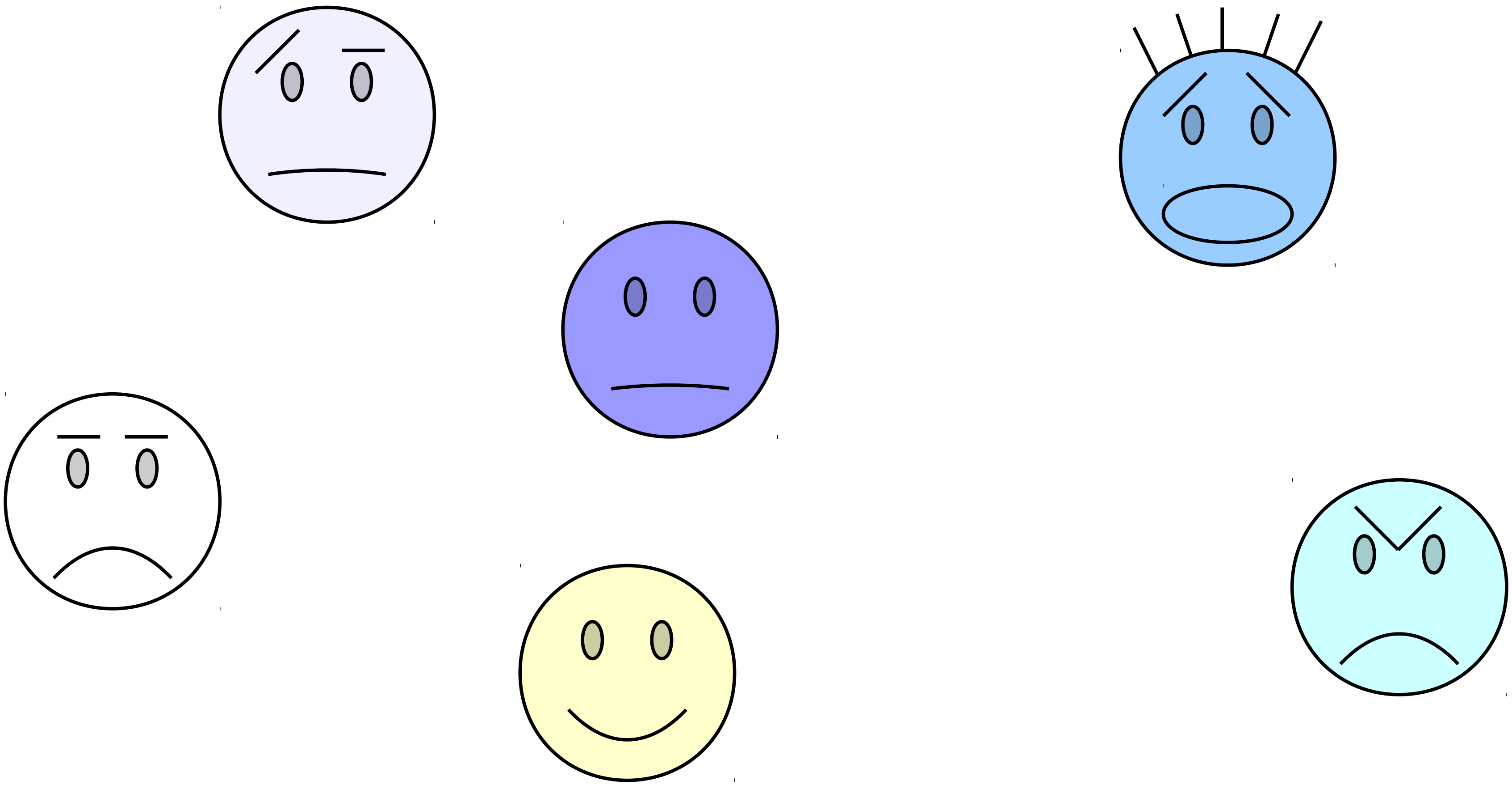
Question: In this world, is the first-order logic statement below true or false?

Respond at pollev.com/zhenglian740



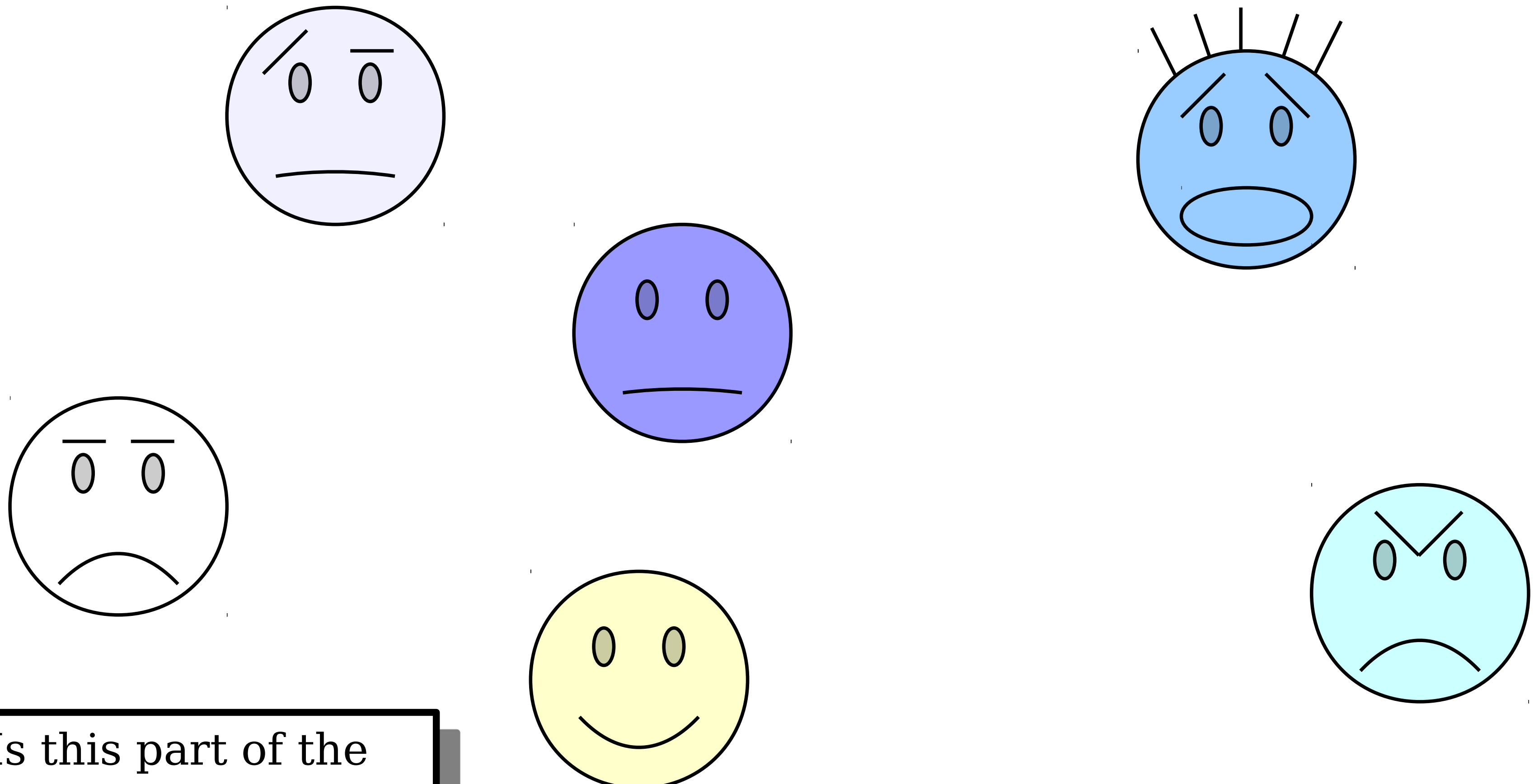
$$(\exists x.\ Smiling(x)) \rightarrow (\exists y.\ WearingHat(y))$$

The Existential Quantifier



$(\exists x.\ Smiling(x)) \rightarrow (\exists y.\ WearingHat(y))$

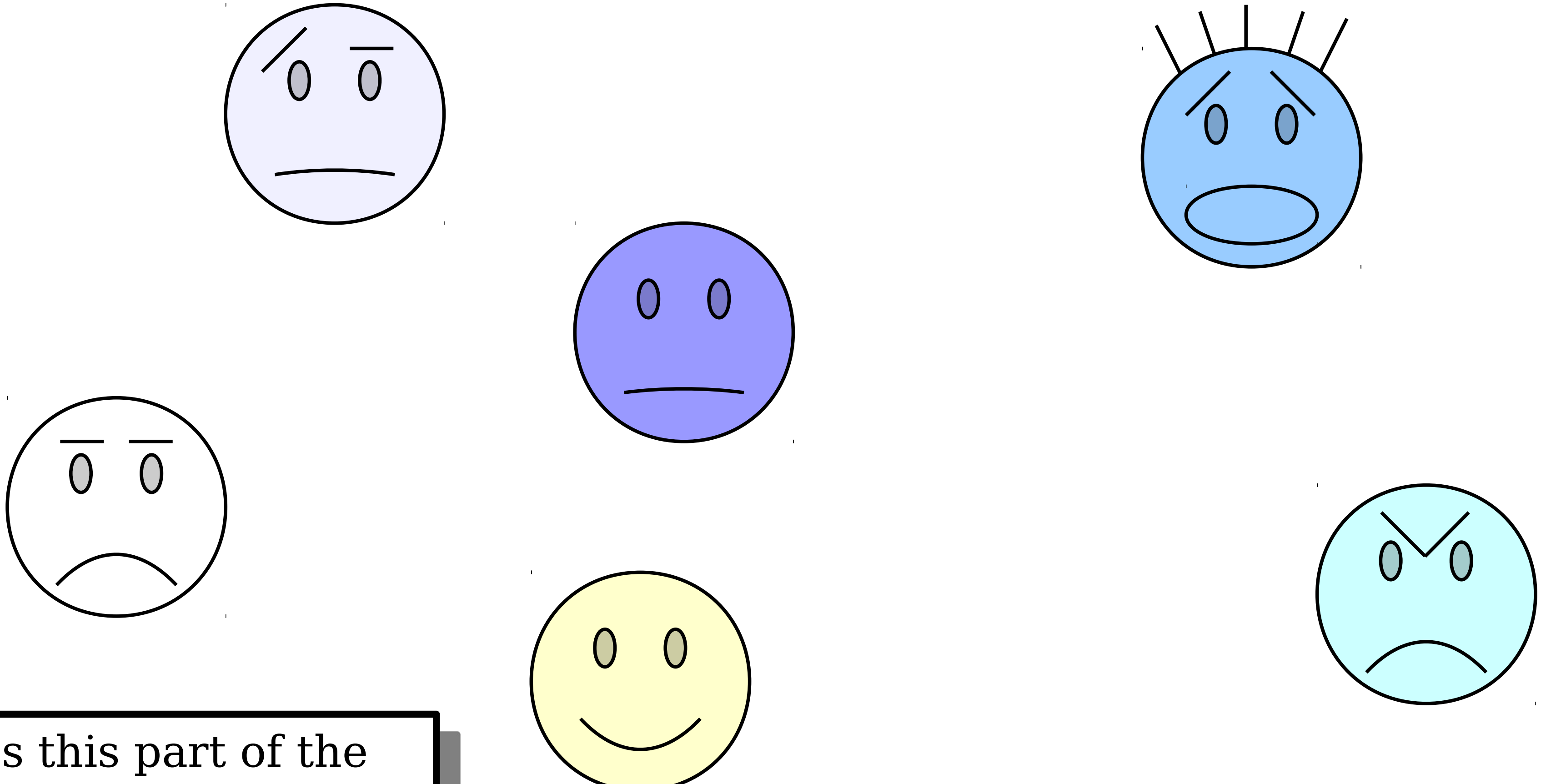
The Existential Quantifier



Is this part of the statement true or false?

$$(\exists x.\ \mathit{Smiling}(x)) \to (\exists y.\ \mathit{WearingHat}(y))$$

The Existential Quantifier



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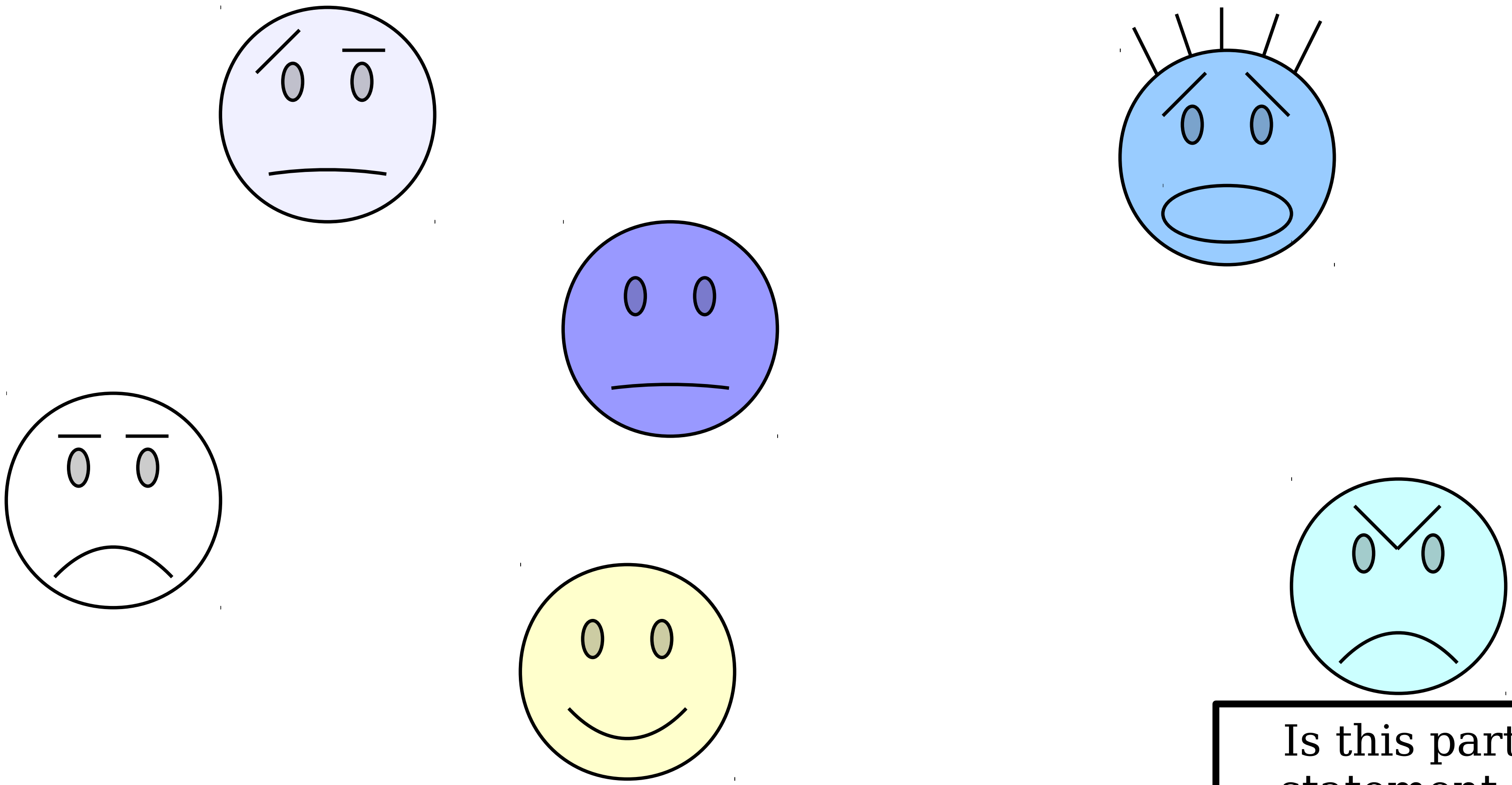

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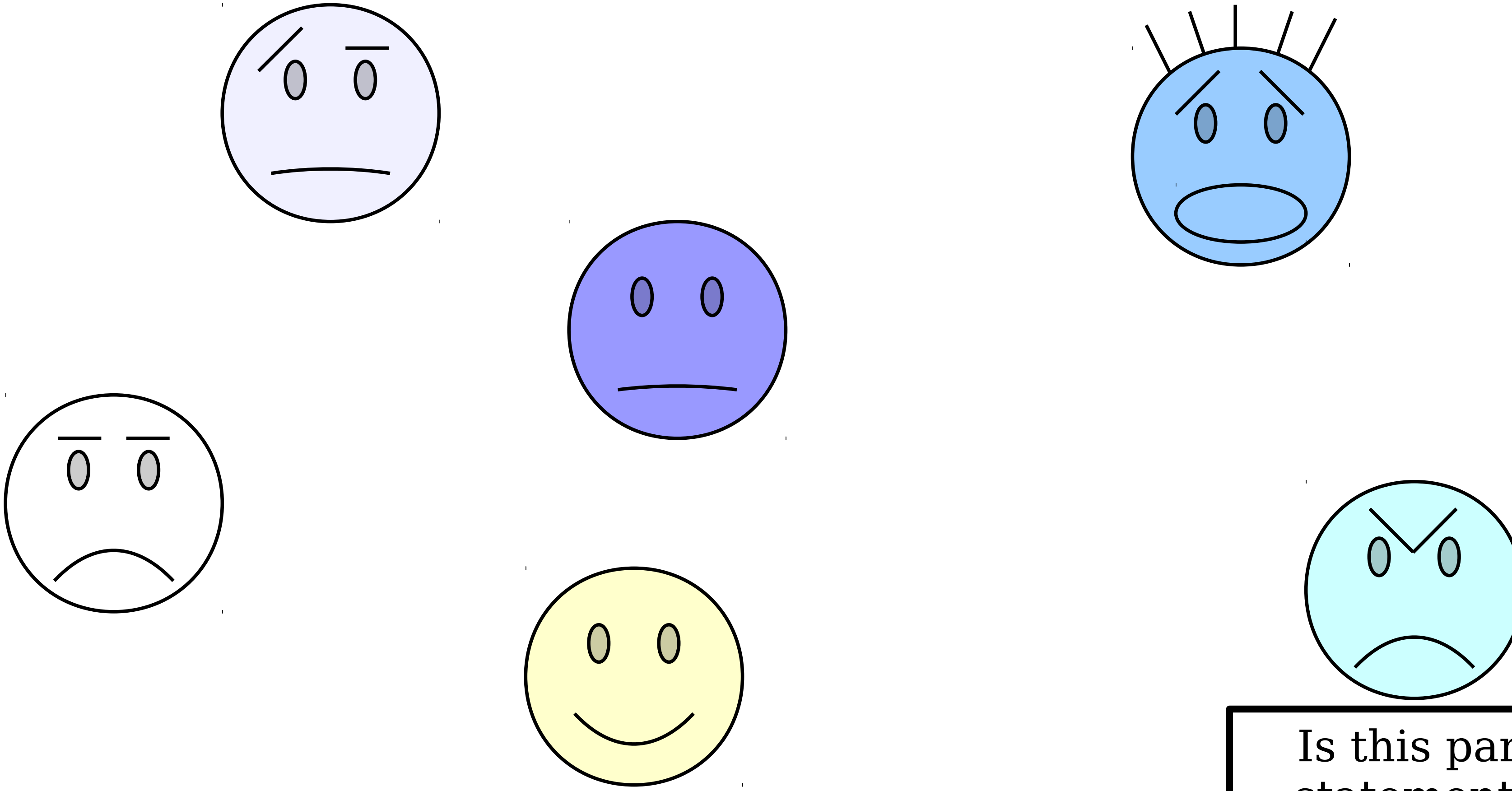
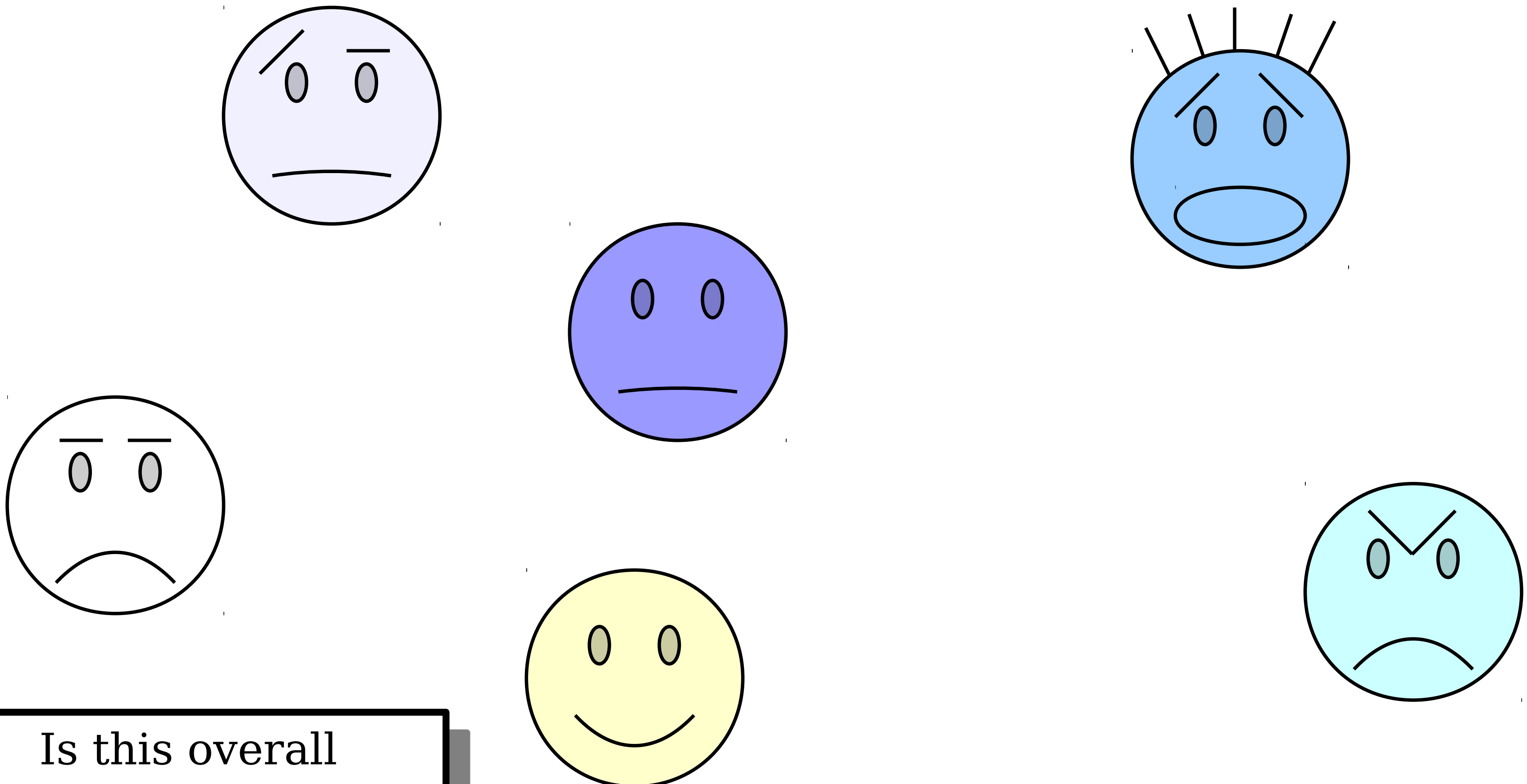


Is this part of the statement true or false?

$(\exists x.\ Smiling(x)) \rightarrow$ ~~$(\exists y.\ WearingHat(y))$~~

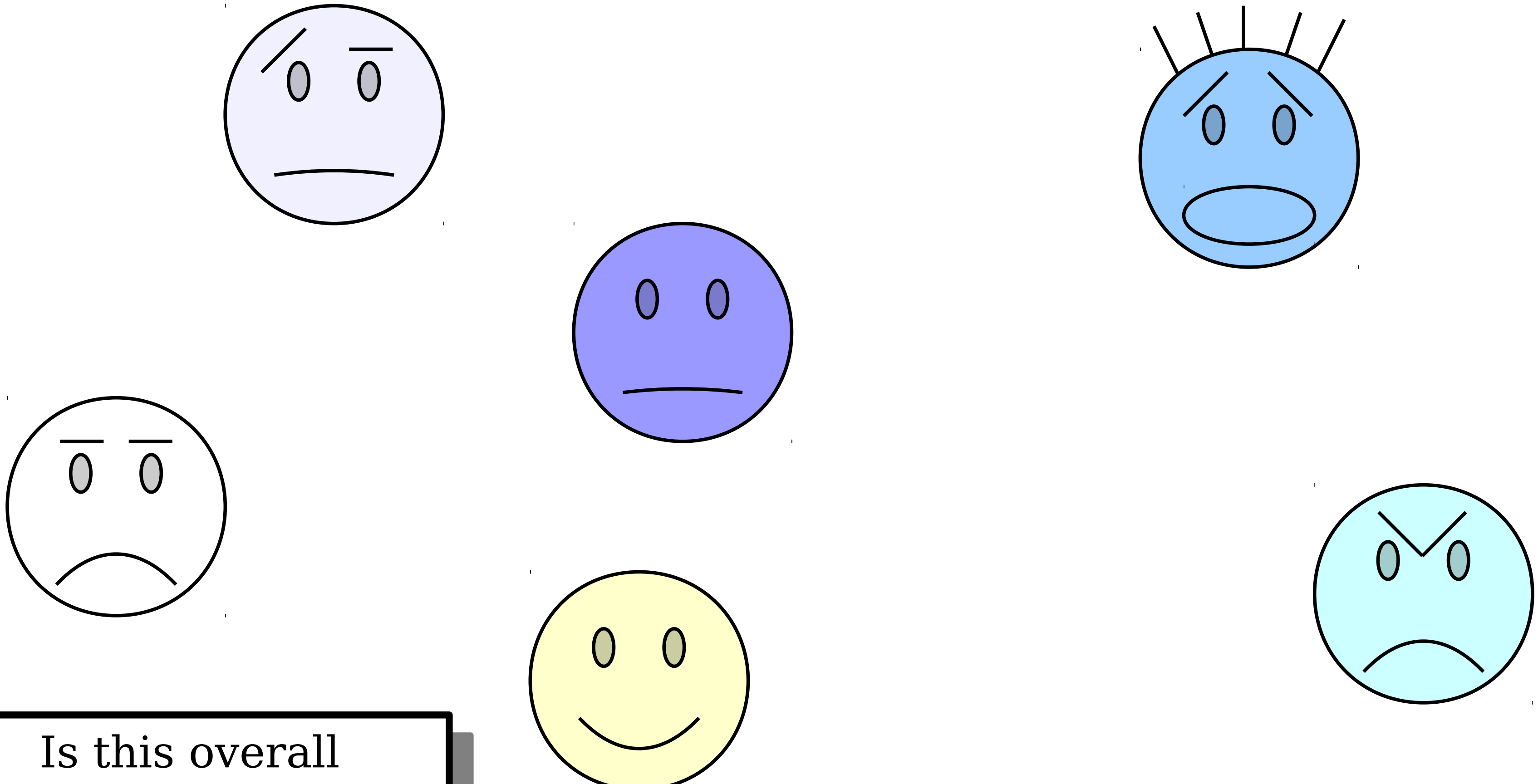
The Existential Quantifier



Is this overall statement true or false?

$(\exists x.\ \mathit{Smiling}(x)) \rightarrow$ ~~$(\exists y.\ \mathit{WearingHat}(y))$~~

The Existential Quantifier



Is this overall statement true or false?

~~$(\exists x.\ \mathit{Smiling}(x)) \rightarrow (\exists y.\ \mathit{WearingHat}(y))$~~

Fun with Edge Cases

$$\exists x.\ Smiling(x)$$

Fun with Edge Cases

Existentially-quantified statements are false in an empty world, since nothing exists, period!

~~$\exists x.\ Smiling(x)$~~

Some Technical Details

Variables and Quantifiers

- Each quantifier has two parts:
  - the variable that is introduced, and
  - the statement that's being quantified.
- The variable introduced is scoped just to the statement being quantified.

$$(\exists x.\ Loves(You, x)) \land (\exists y.\ Loves(y, You))$$

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$$(\exists x.\ Loves(You, x)) \land (\exists y.\ Loves(y, You))$$

The variable ***x*** just lives here.

The variable ***y*** just lives here.

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  - the variable that is introduced, and
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- The variable introduced is scoped just to the statement being quantified.

$$(\exists x.\ Loves(You,\ x)) \land (\exists x.\ Loves(x,\ You))$$

The variable ***x*** just lives here.

A different variable, also named ***x***, just lives here.

Operator Precedence (Again)

- When writing out a formula in first-order logic, quantifiers have precedence just below ¬.
- The statement

$$\exists x.\ P(x) \land R(x) \land Q(x)$$

  is parsed like this:

$$⚠ \qquad (\exists x.\ P(x)) \ \land\ (R(x) \land Q(x)) \qquad ⚠$$

- This is syntactically invalid because the variable x is out of scope in the back half of the formula.
- To ensure that x is properly quantified, explicitly put parentheses around the region you want to quantify:

$$\exists x.\ (P(x) \land R(x) \land Q(x))$$

“For any natural number n,
n is even if and only if n^2 is even”

"For any natural number n,
n is even if and only if n^2 is even"

$$\forall n.\ (n \in \mathbb{N} \to (Even(n) \leftrightarrow Even(n^2)))$$

"For any natural number *n*,
n is even if and only if n^2 is even"

$$\forall n.\ (n \in \mathbb{N} \to (Even(n) \leftrightarrow Even(n^2)))$$

∀ is the ***universal quantifier*** and says "for any choice of *n*, the following is true."

The Universal Quantifier

- A statement of the form

  $$\forall x.\ \textit{some-formula}$$

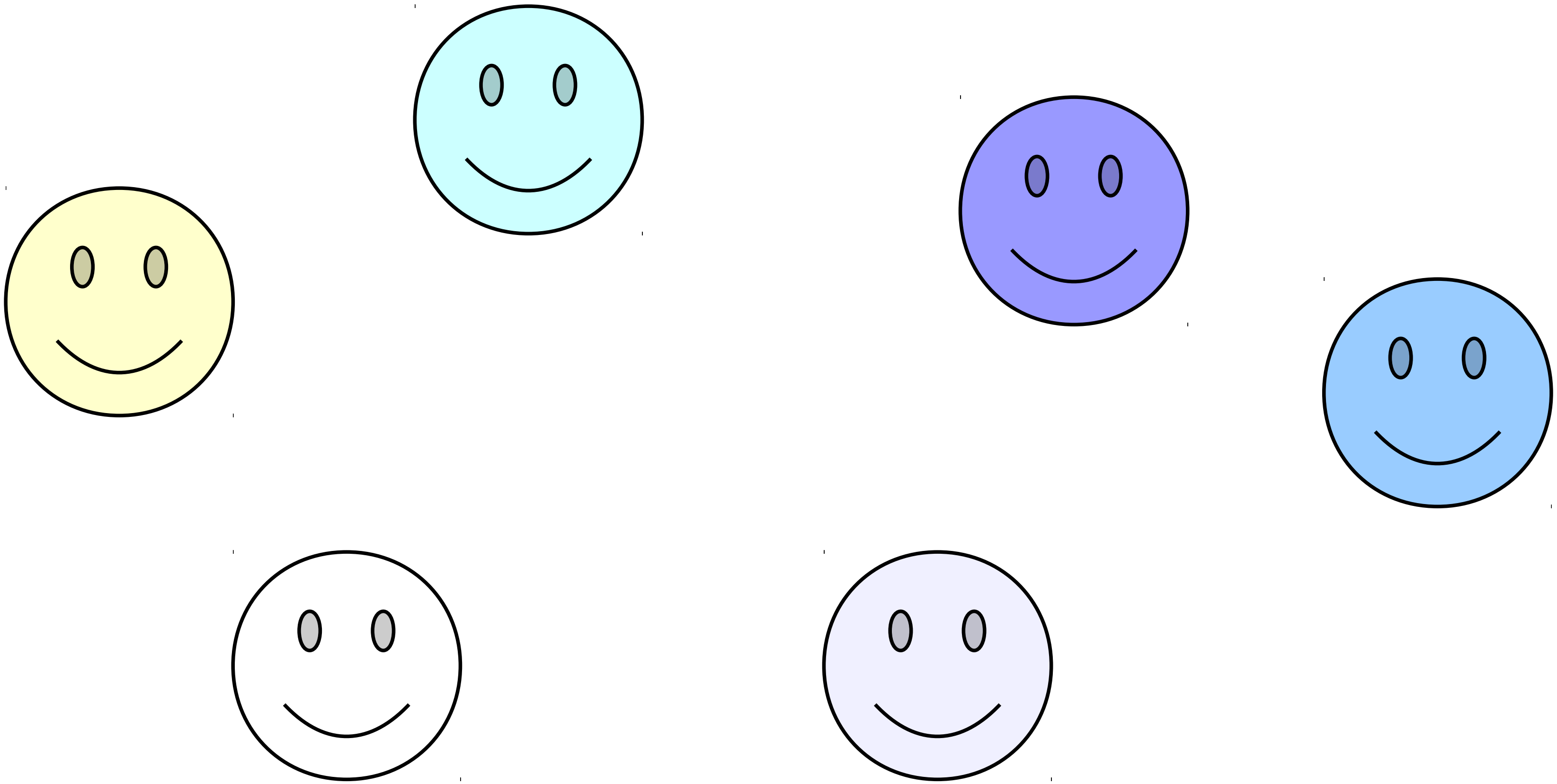
  is true if, for every choice of x, the statement ***some-formula*** is true when x is plugged into it.
- Examples:

  $$\forall p.\ (Puppy(p) \to Cute(p))$$

  $$\forall a.\ (EatsPlants(a) \lor EatsAnimals(a))$$

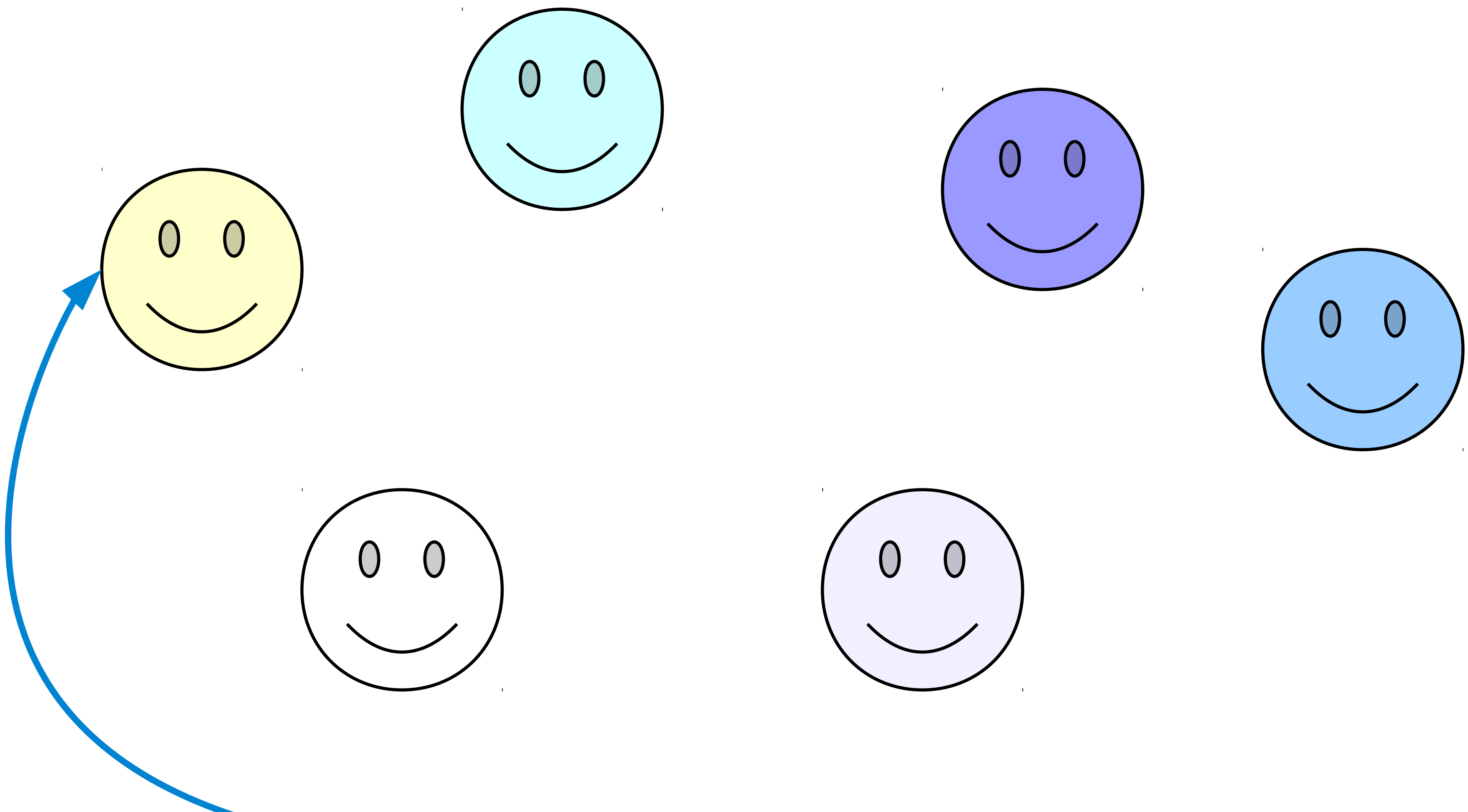
  $$Tallest(SultanK\ddot{o}sen) \to$$
  $$\quad \forall x.\ (SultanK\ddot{o}sen \neq x \to ShorterThan(x, SultanK\ddot{o}sen))$$

The Universal Quantifier



$$\forall x.\ Smiling(x)$$

The Universal Quantifier



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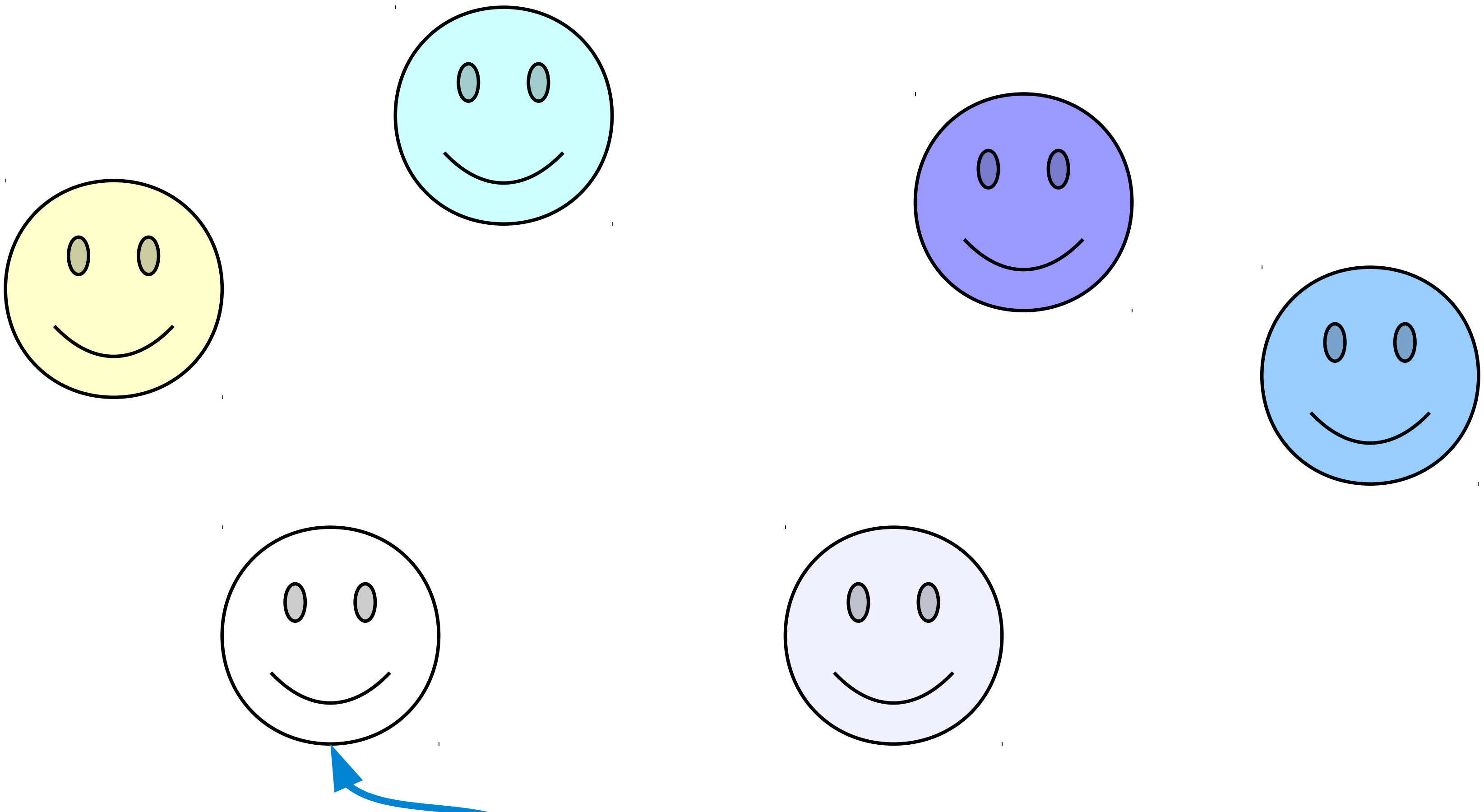

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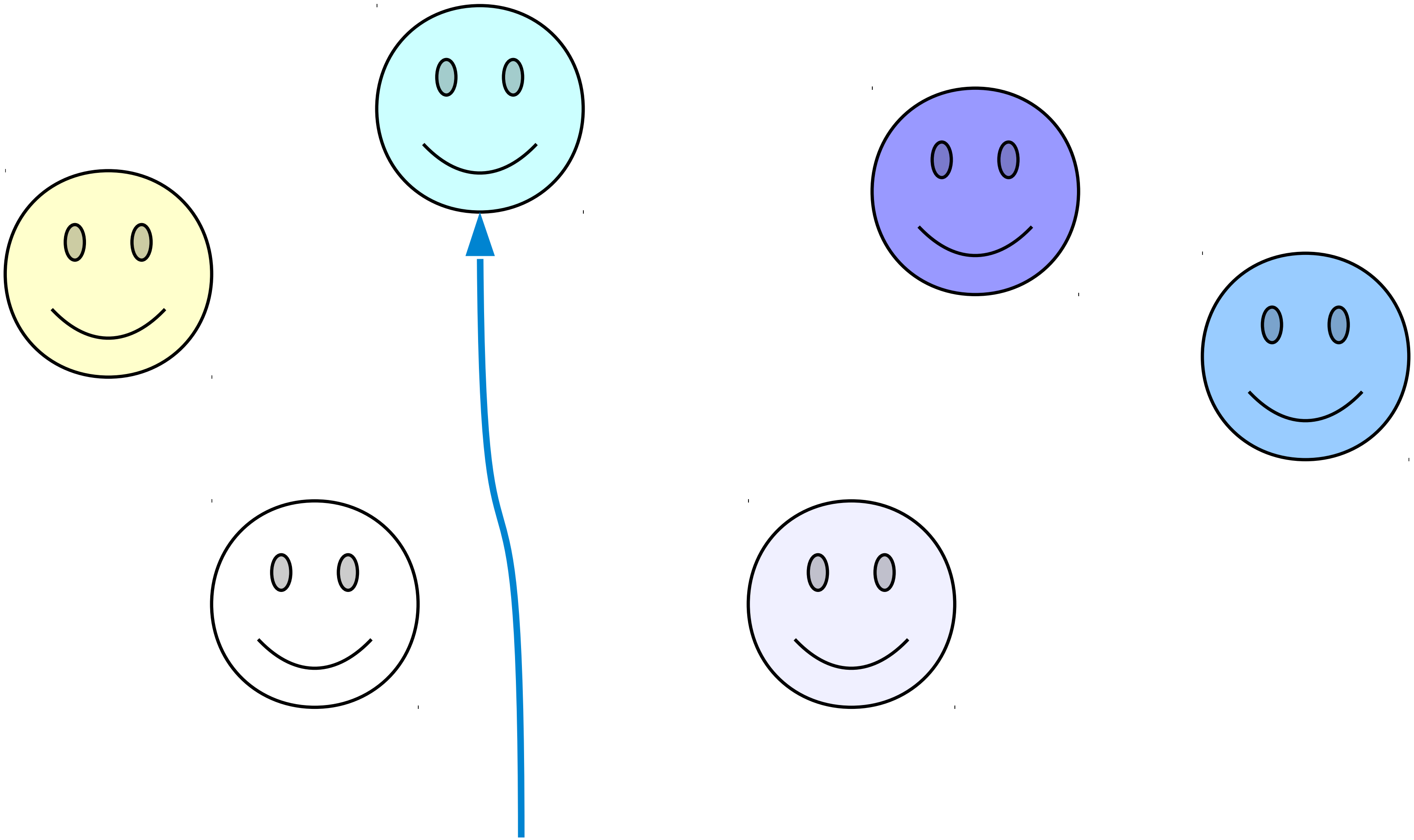

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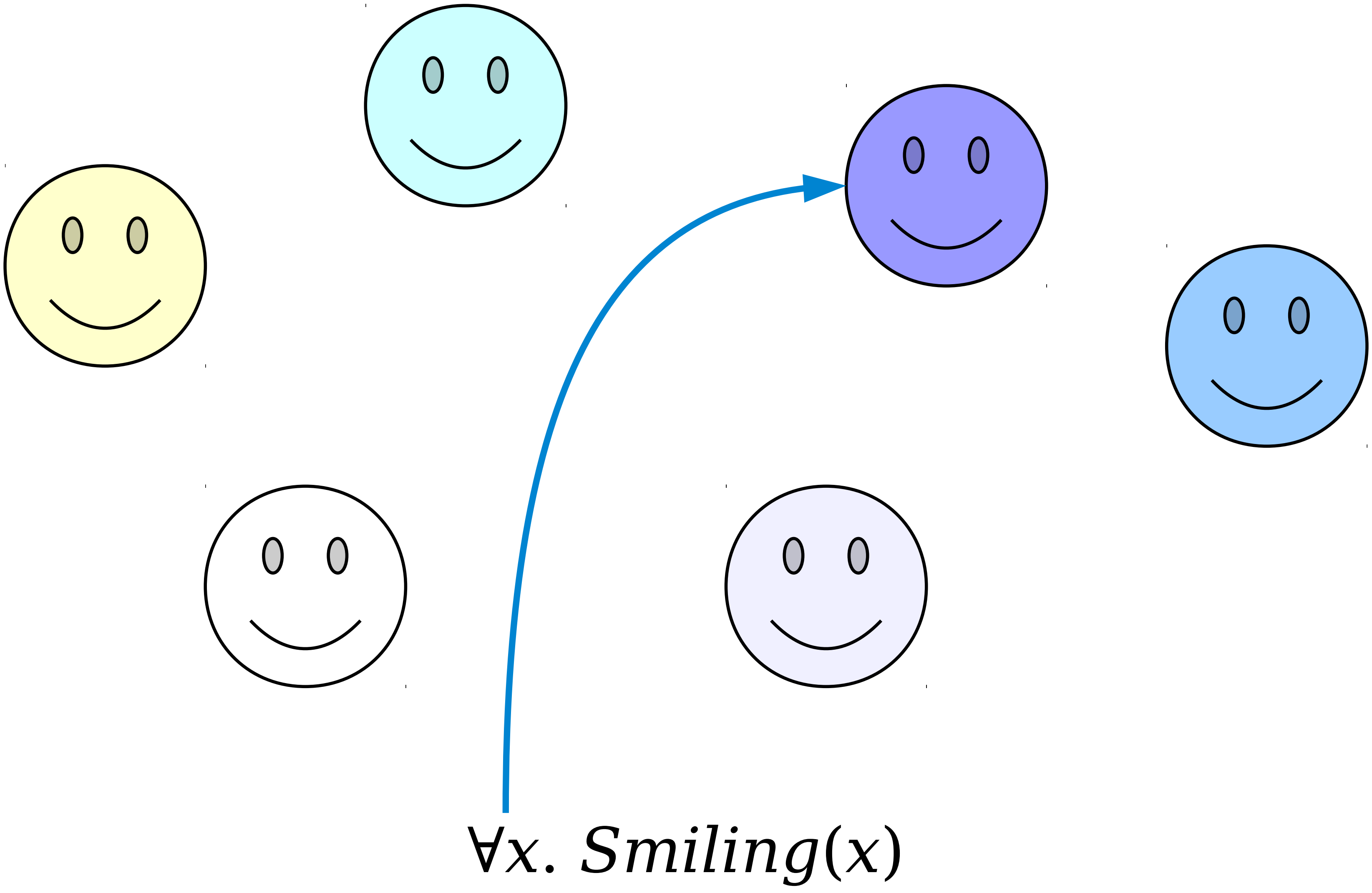

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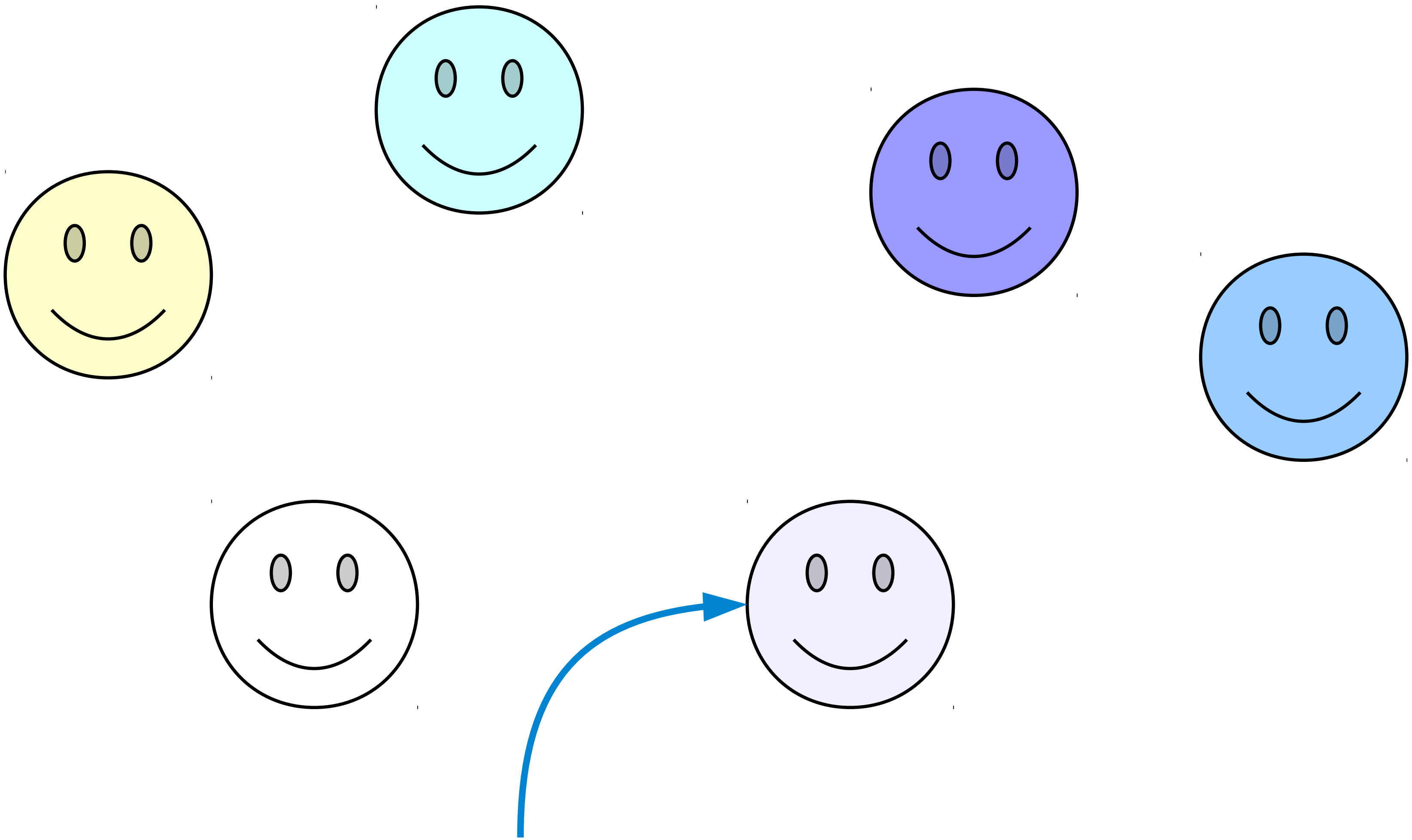

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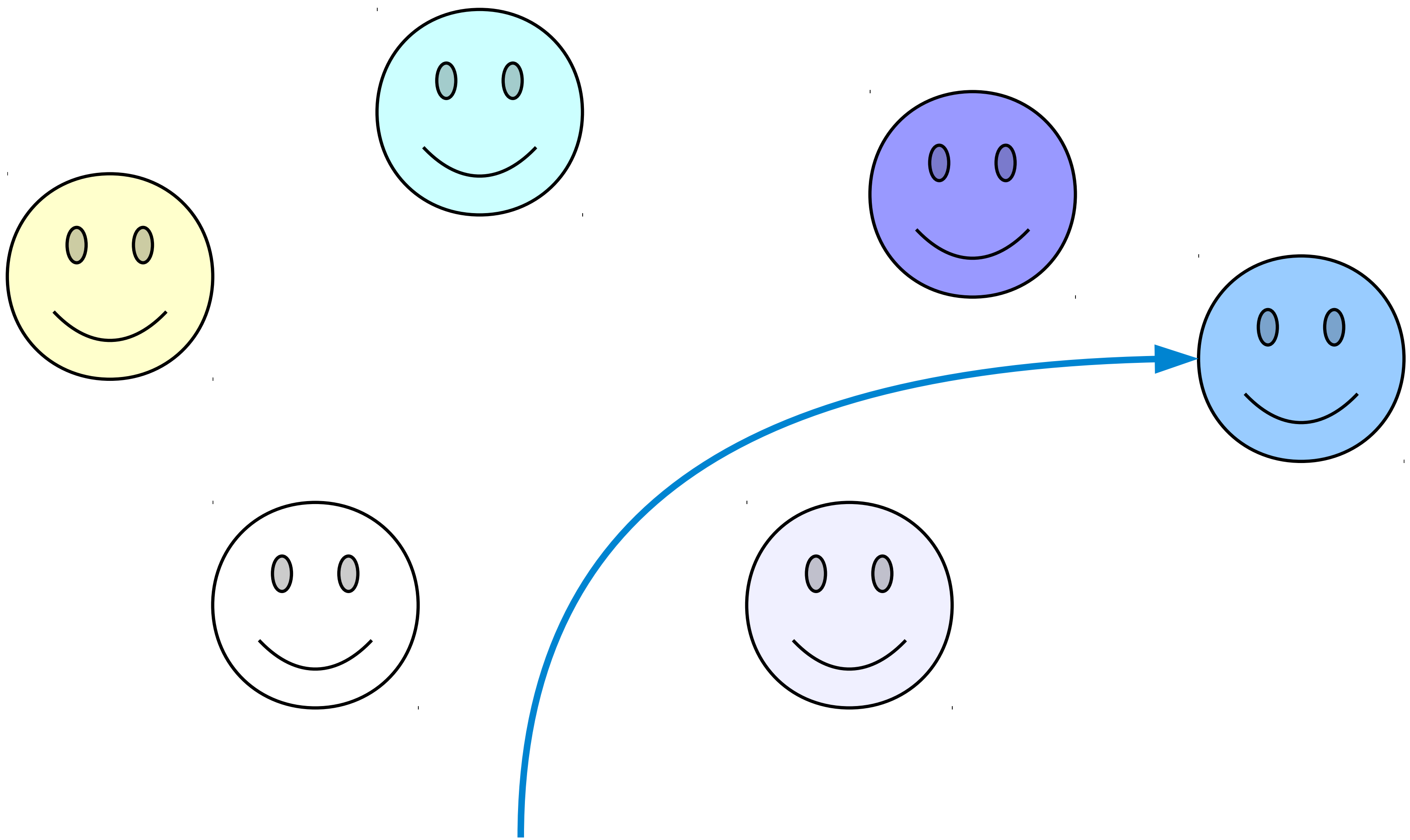

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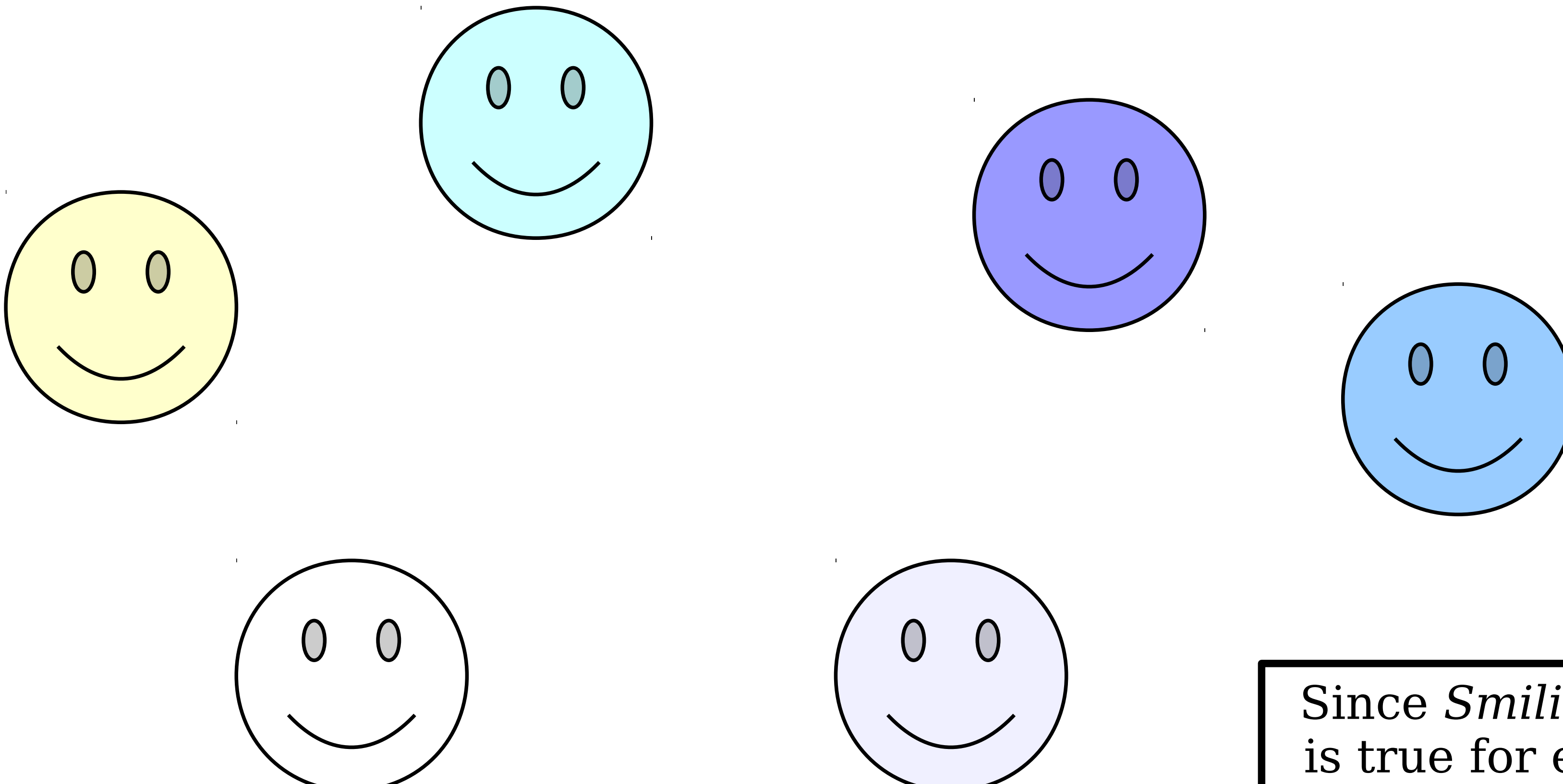
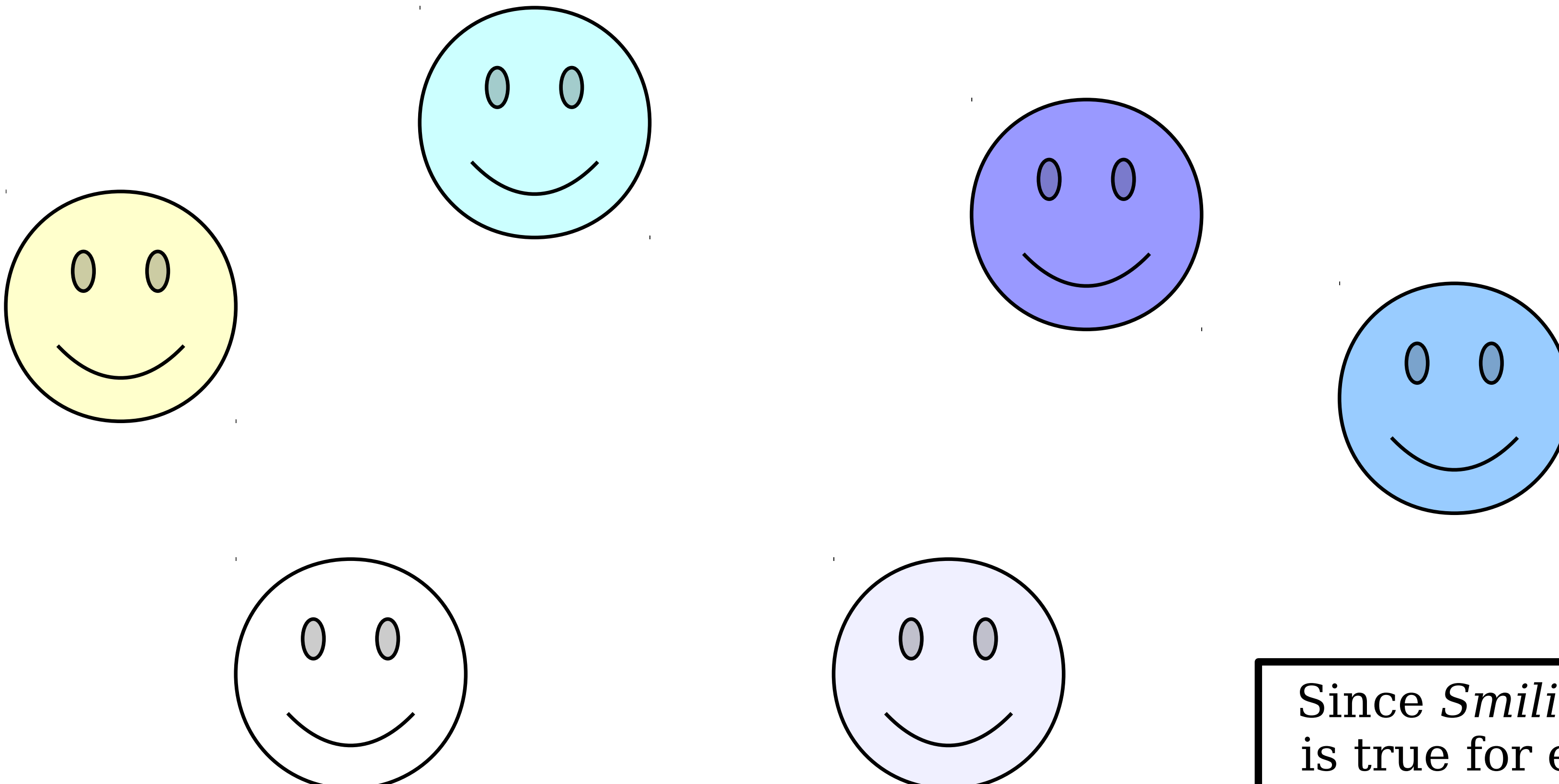
The Universal Quantifier



Since $Smiling(x)$ is true for every choice of x, this statement evaluates to true.

$$\forall x.\ Smiling(x)$$

The Universal Quantifier



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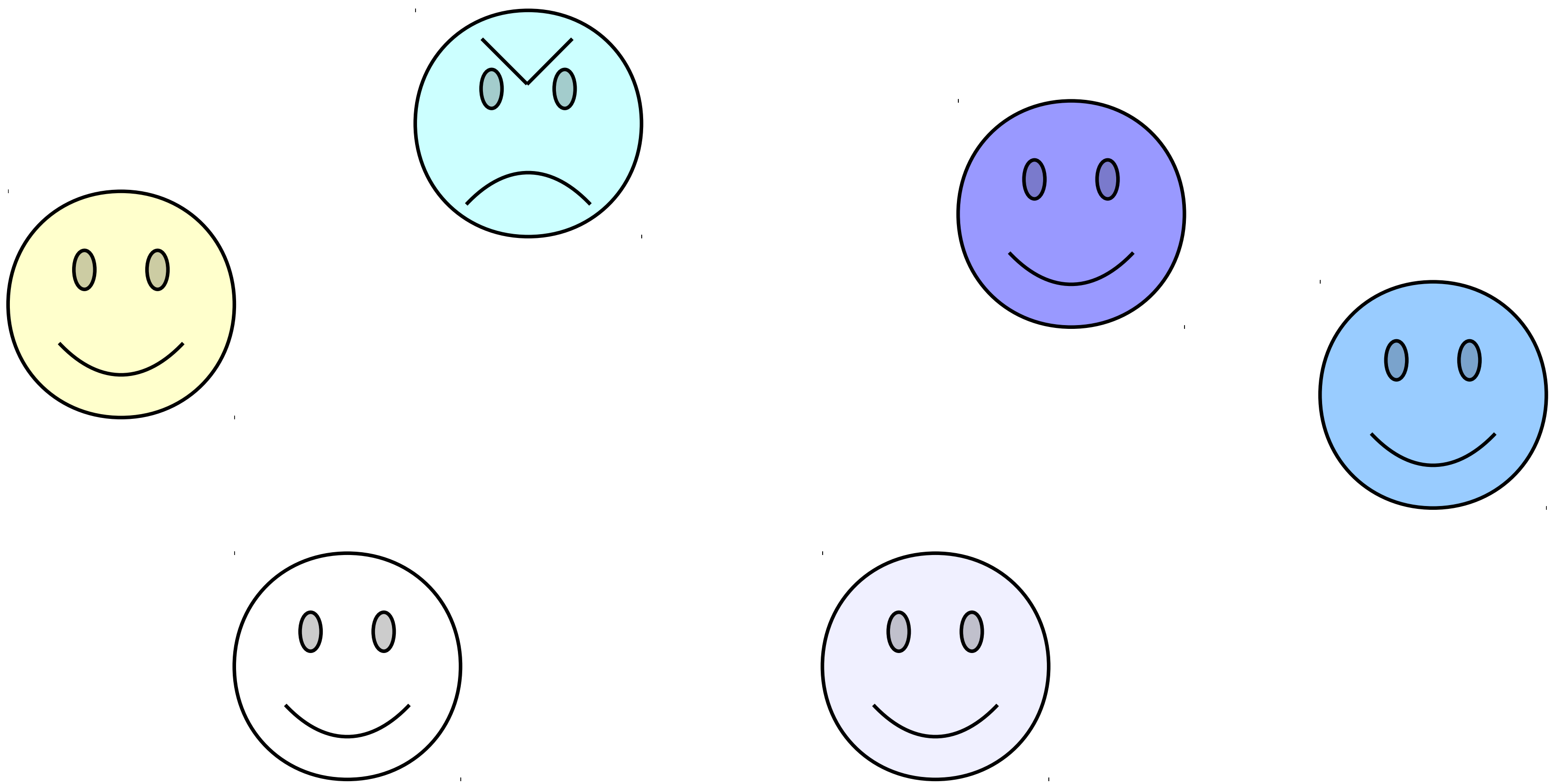

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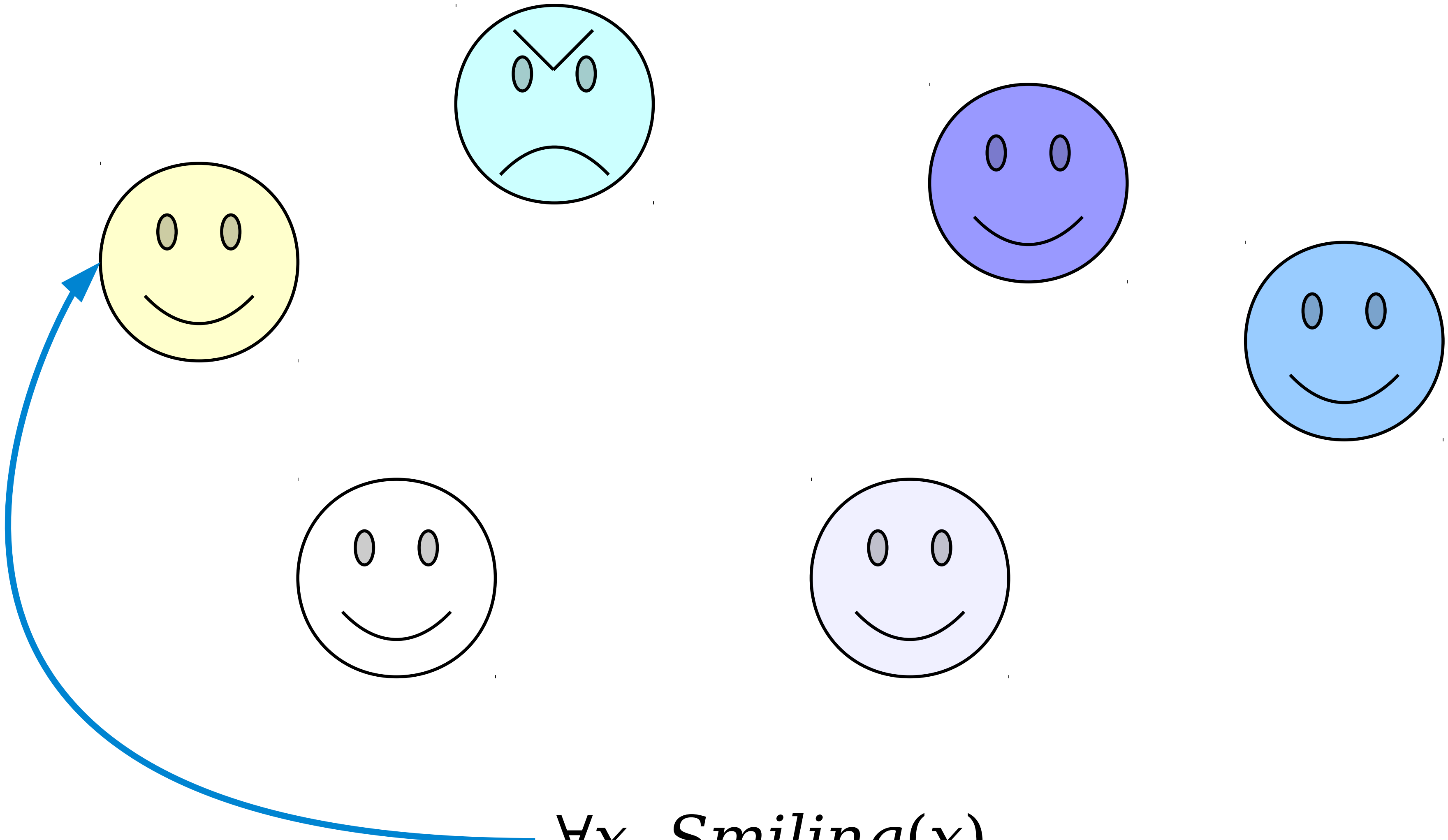
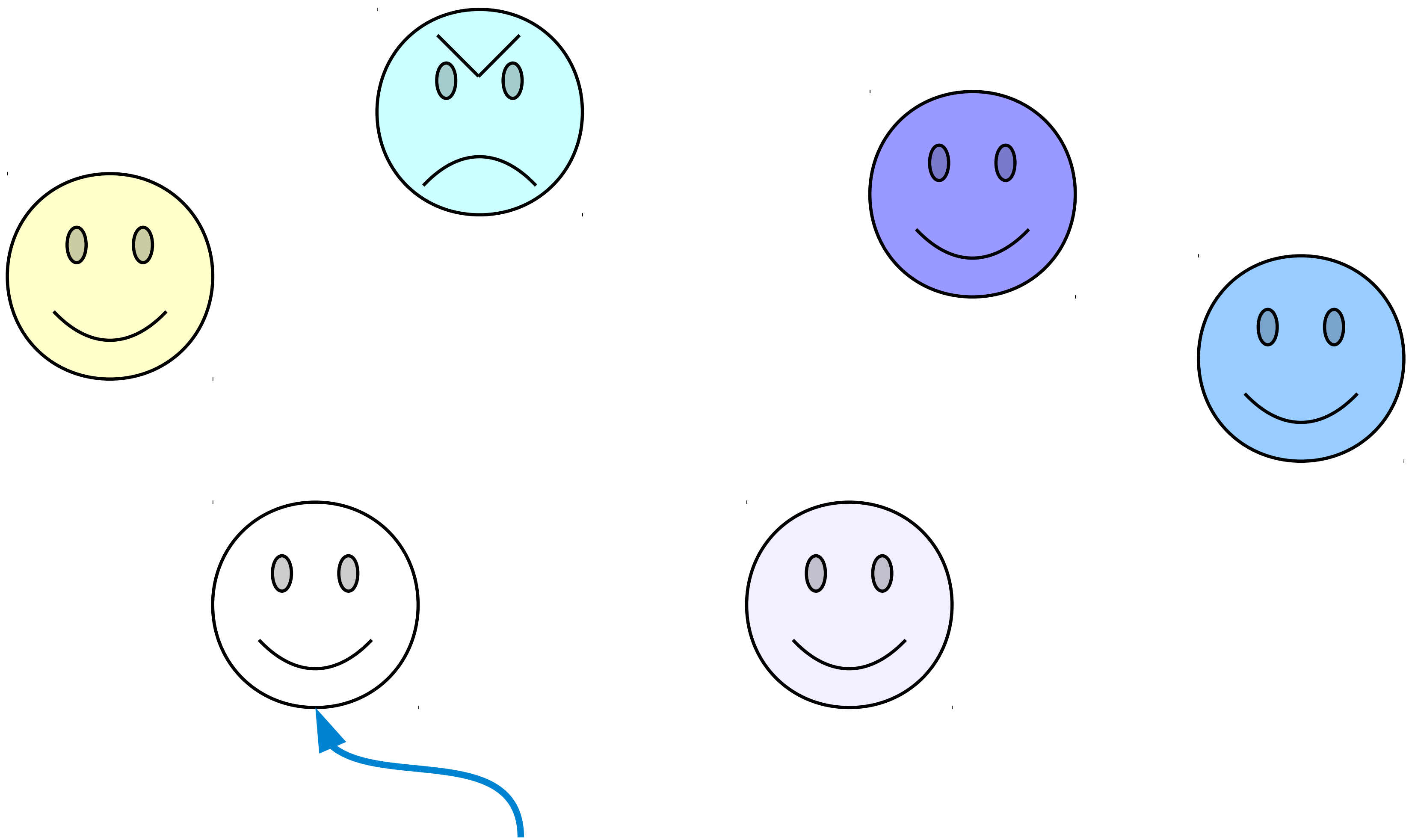


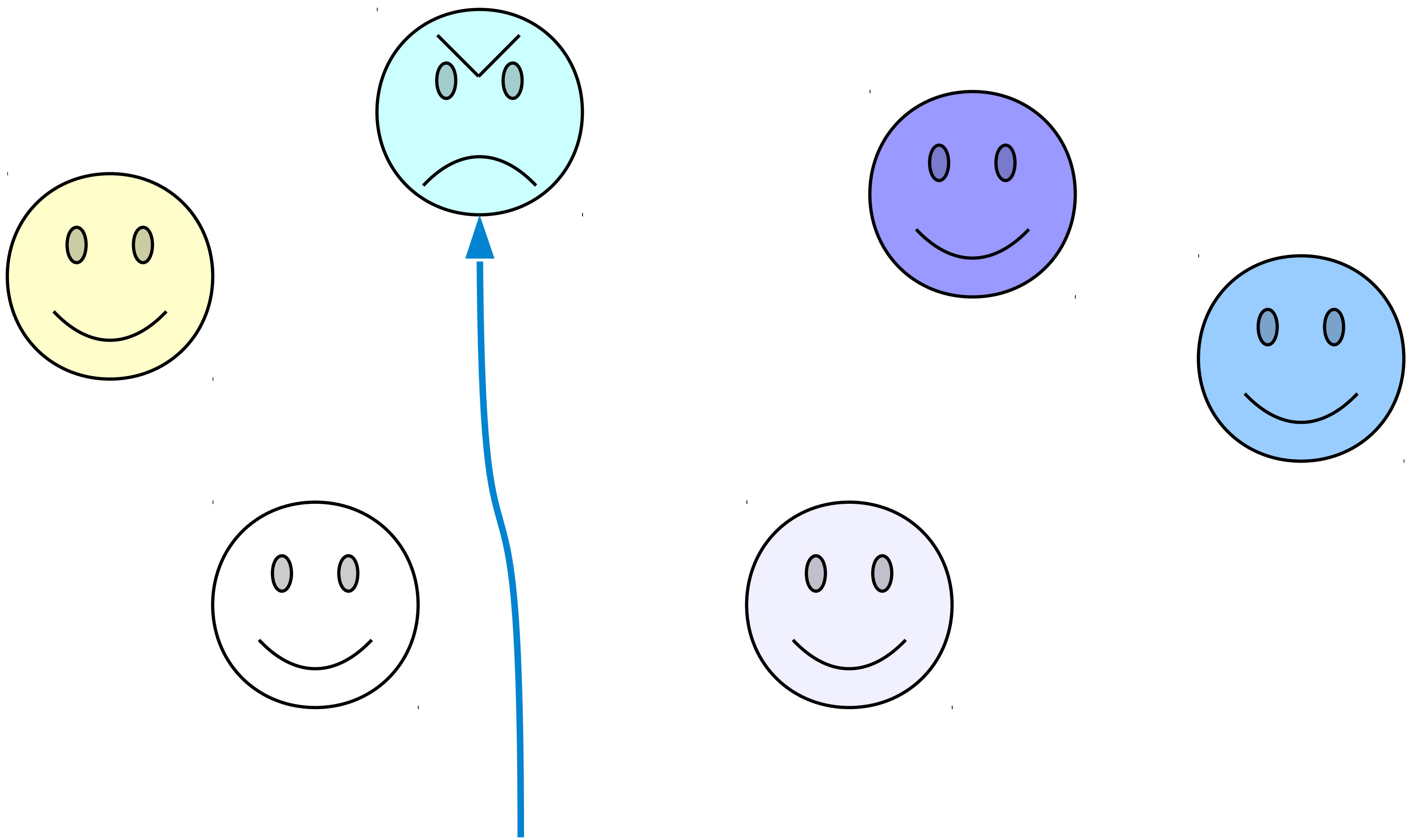
$\forall x.\ Smiling(x)$

The Universal Quantifier



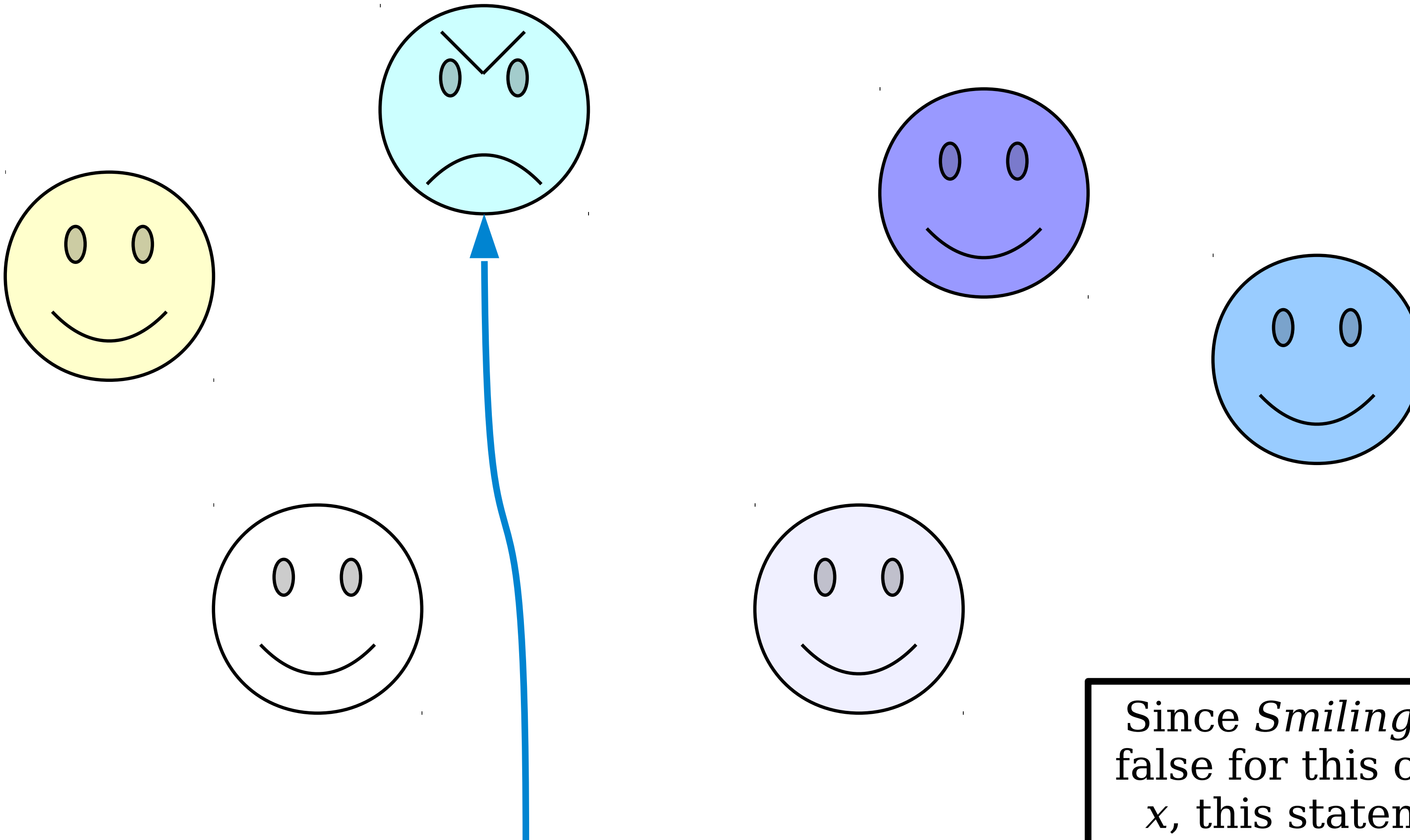
$$\forall x.\ Smiling(x)$$

The Universal Quantifier



$\forall x.\ Smiling(x)$

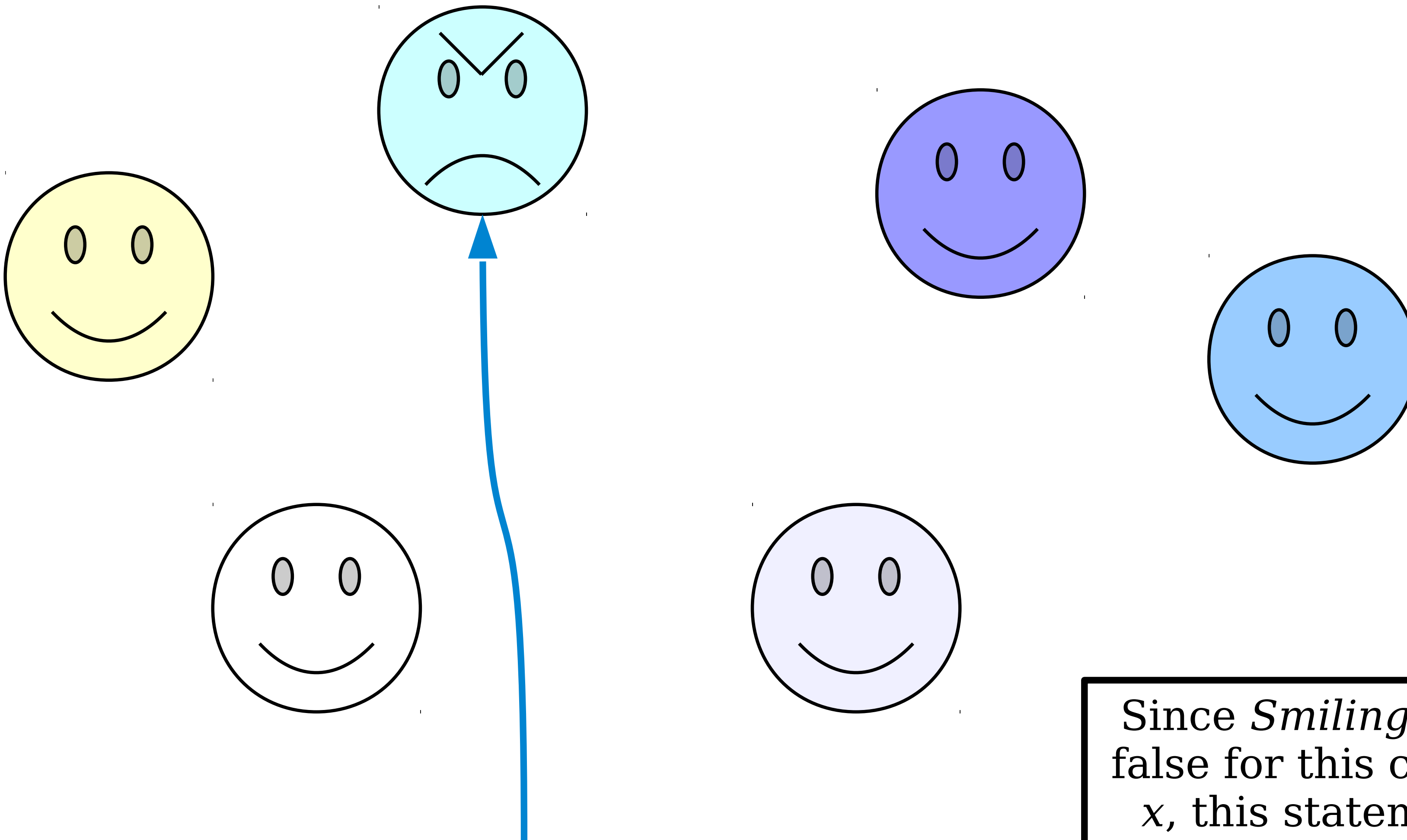
The Universal Quantifier



$\forall x.\ Smiling(x)$

Since *Smiling*(*x*) is false for this choice *x*, this statement evaluates to false.

The Universal Quantifier



~~$\forall x.\ Smiling(x)$~~

Since $Smiling(x)$ is false for this choice x, this statement evaluates to false.

Question: In this world, is the first-order logic statement below true or false?

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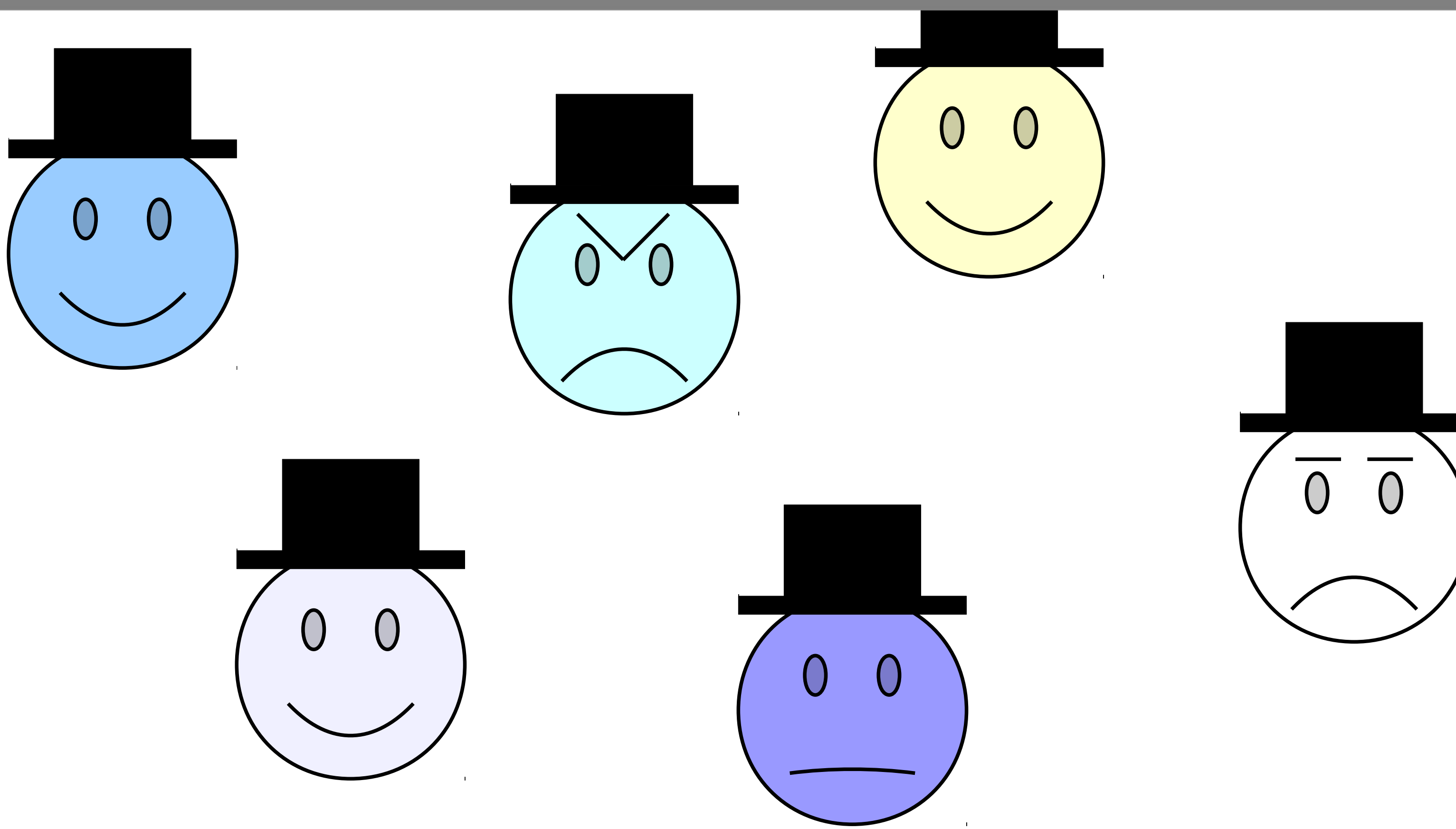
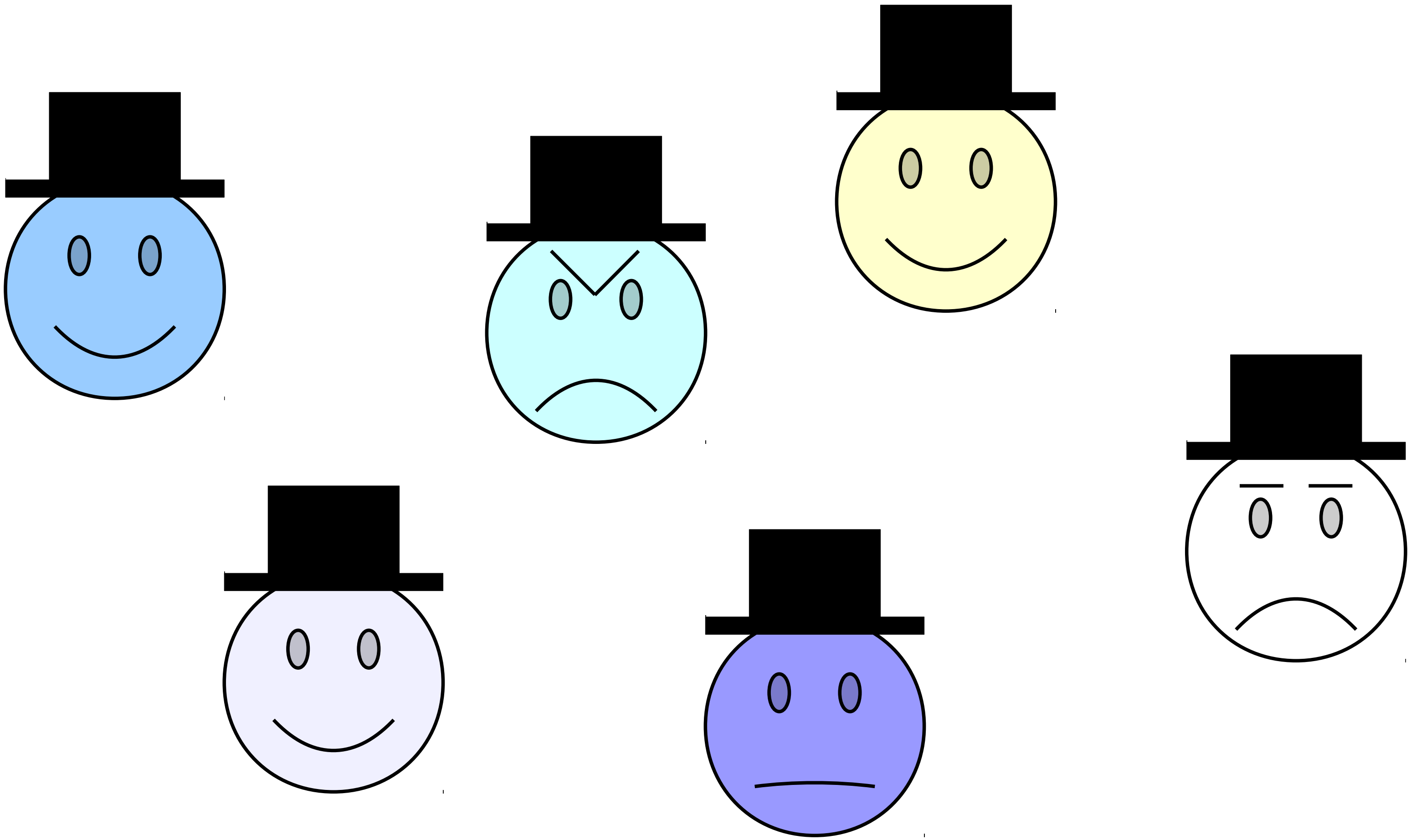


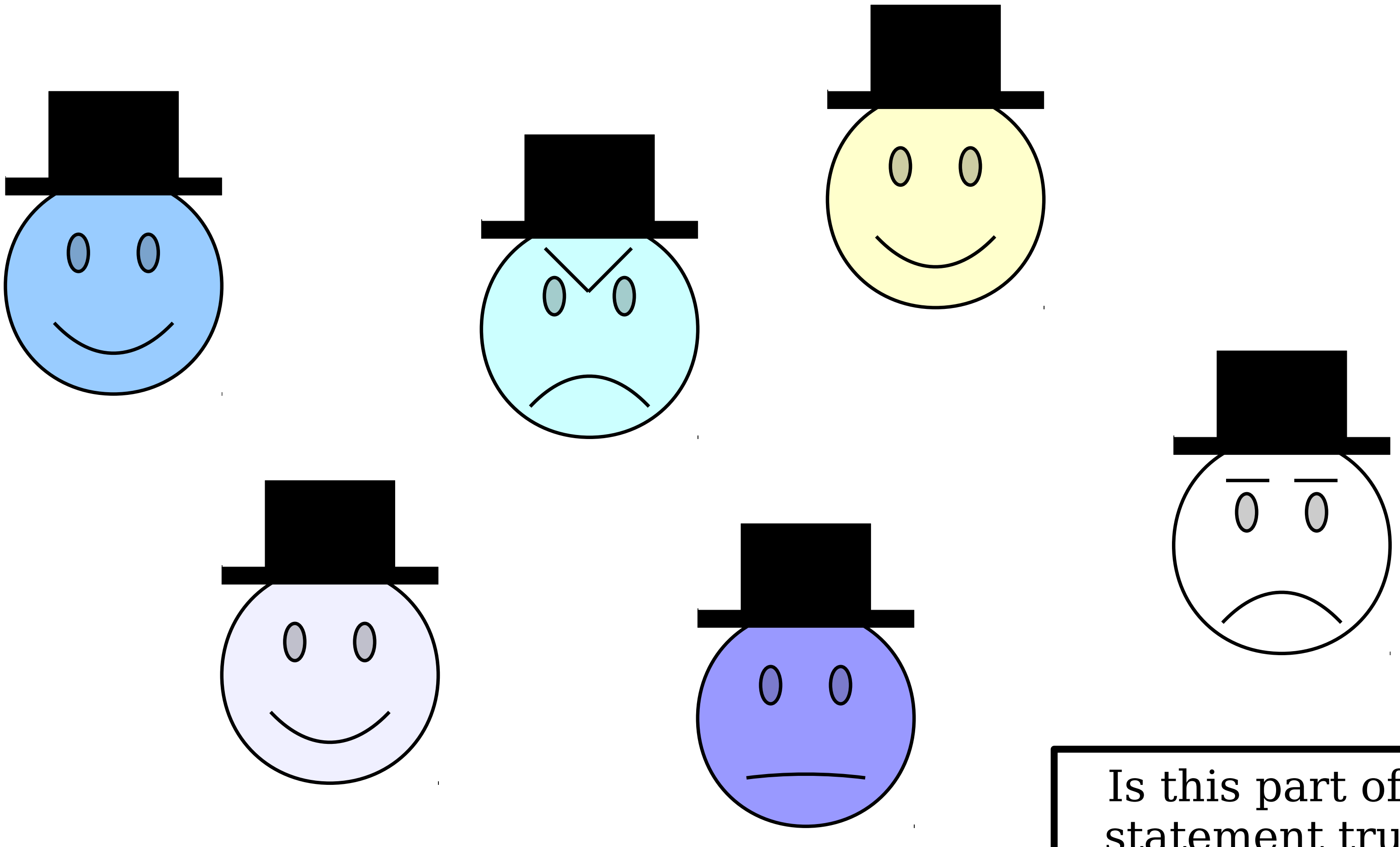
$$(\forall x.\ Smiling(x)) \rightarrow (\forall y.\ WearingHat(y))$$

The Universal Quantifier



$(\forall x.\ Smiling(x)) \rightarrow (\forall y.\ WearingHat(y))$

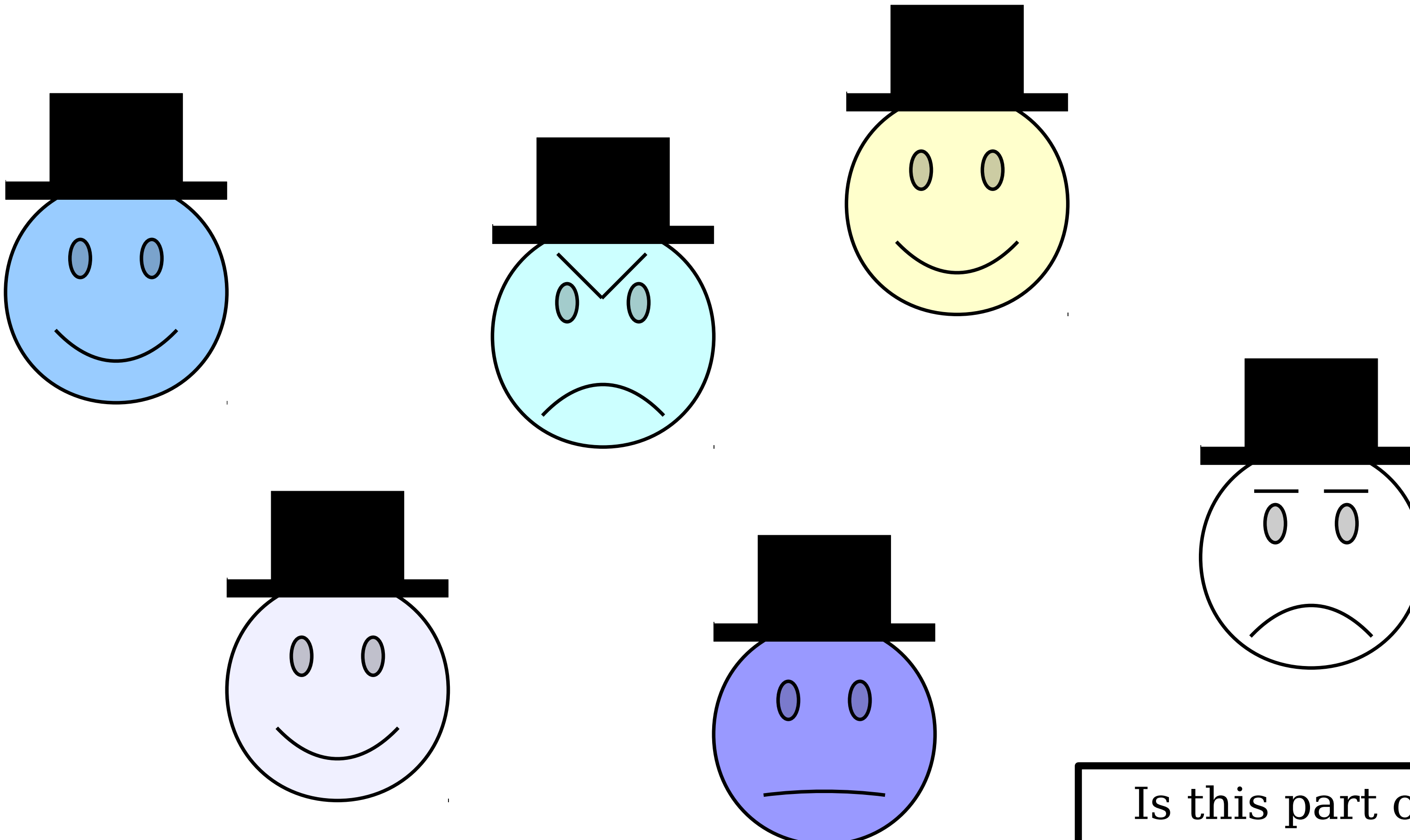
The Universal Quantifier



Is this part of the statement true or false?

$(\forall x.\ Smiling(x)) \rightarrow (\forall y.\ WearingHat(y))$

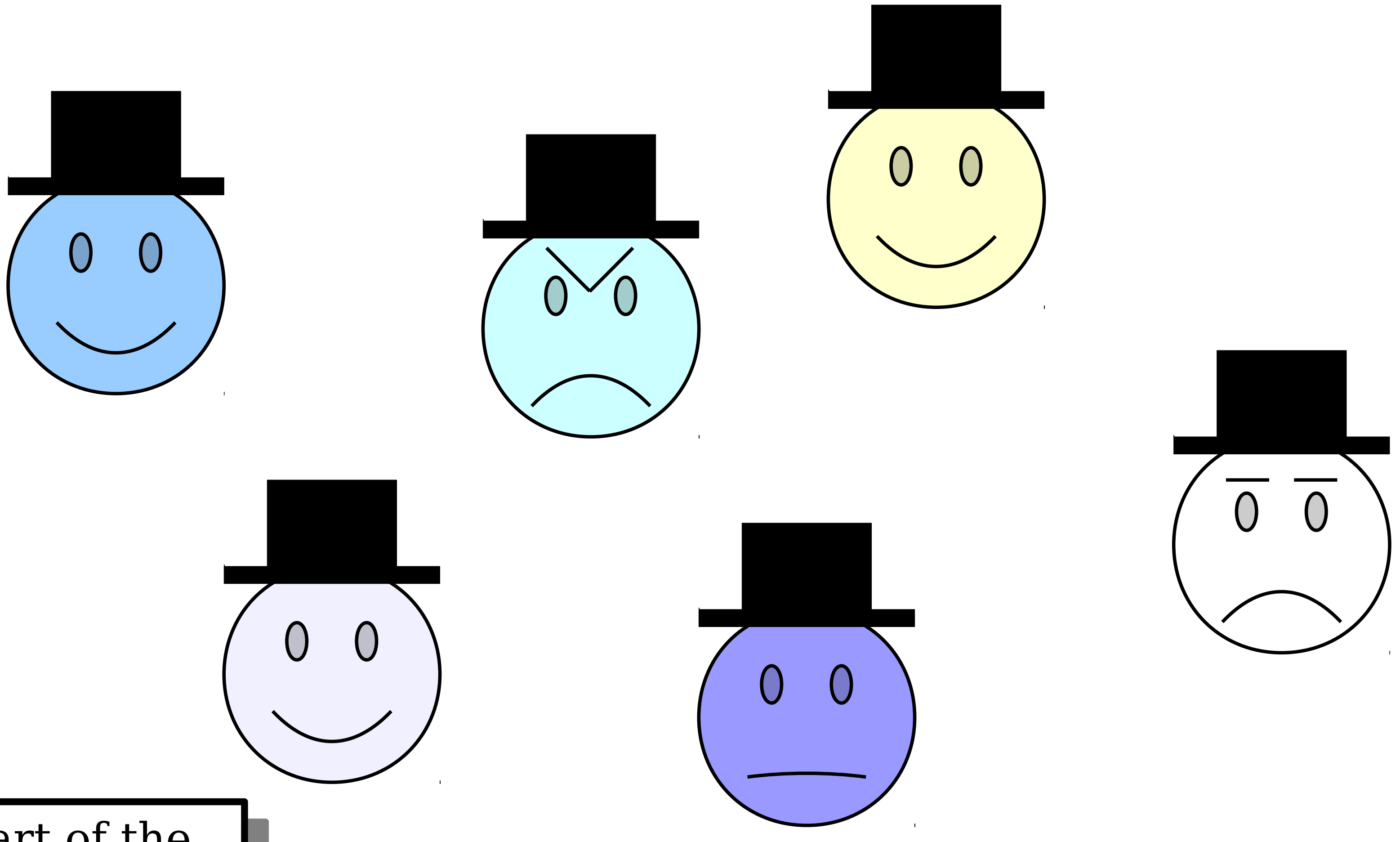
The Universal Quantifier



Is this part of the statement true or false?

$$(\forall x.\ Smiling(x)) \to (\forall y.\ WearingHat(y))$$

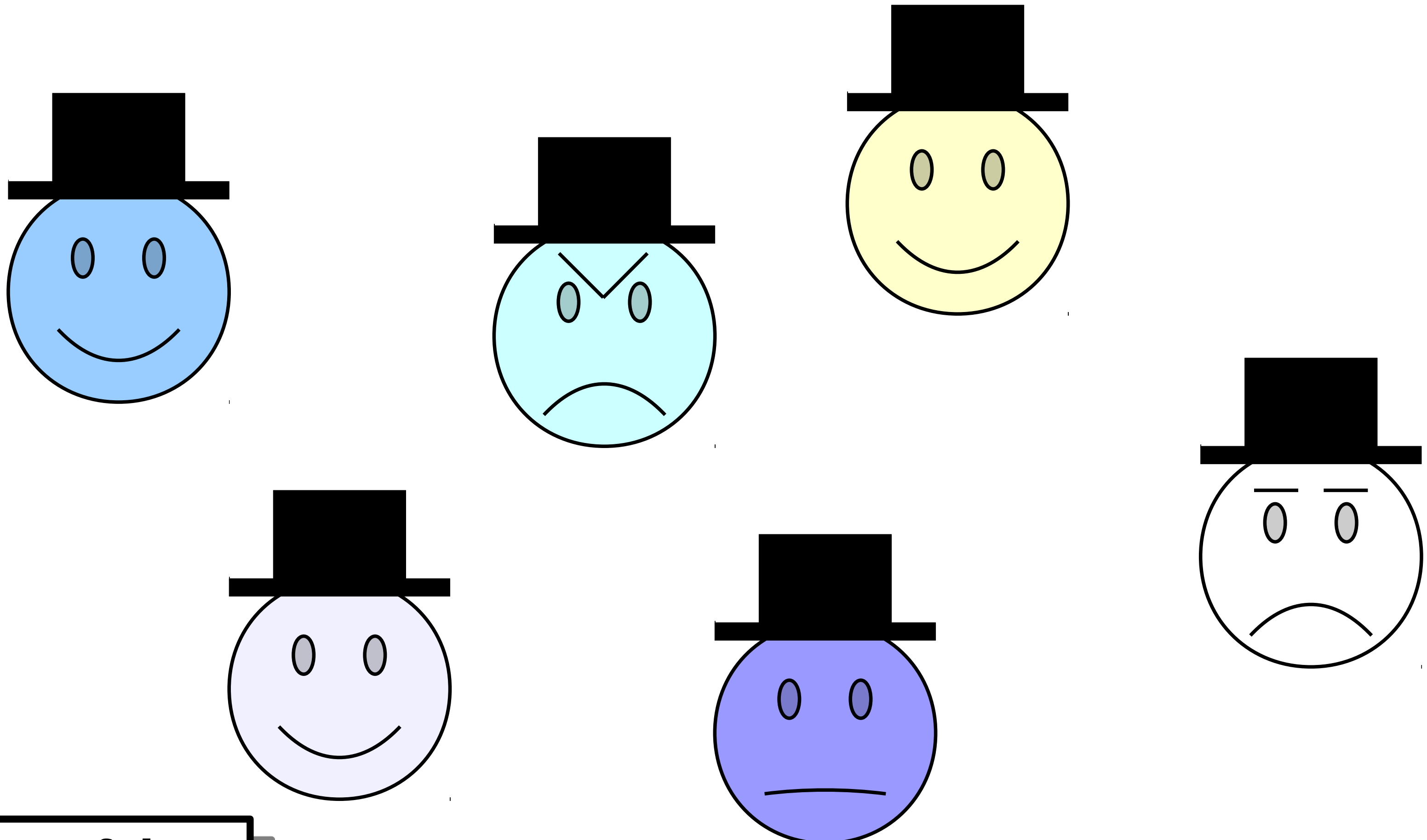
The Universal Quantifier



Is this part of the statement true or false?

$(\forall x.\ Smiling(x)) \to (\forall y.\ WearingHat(y))$

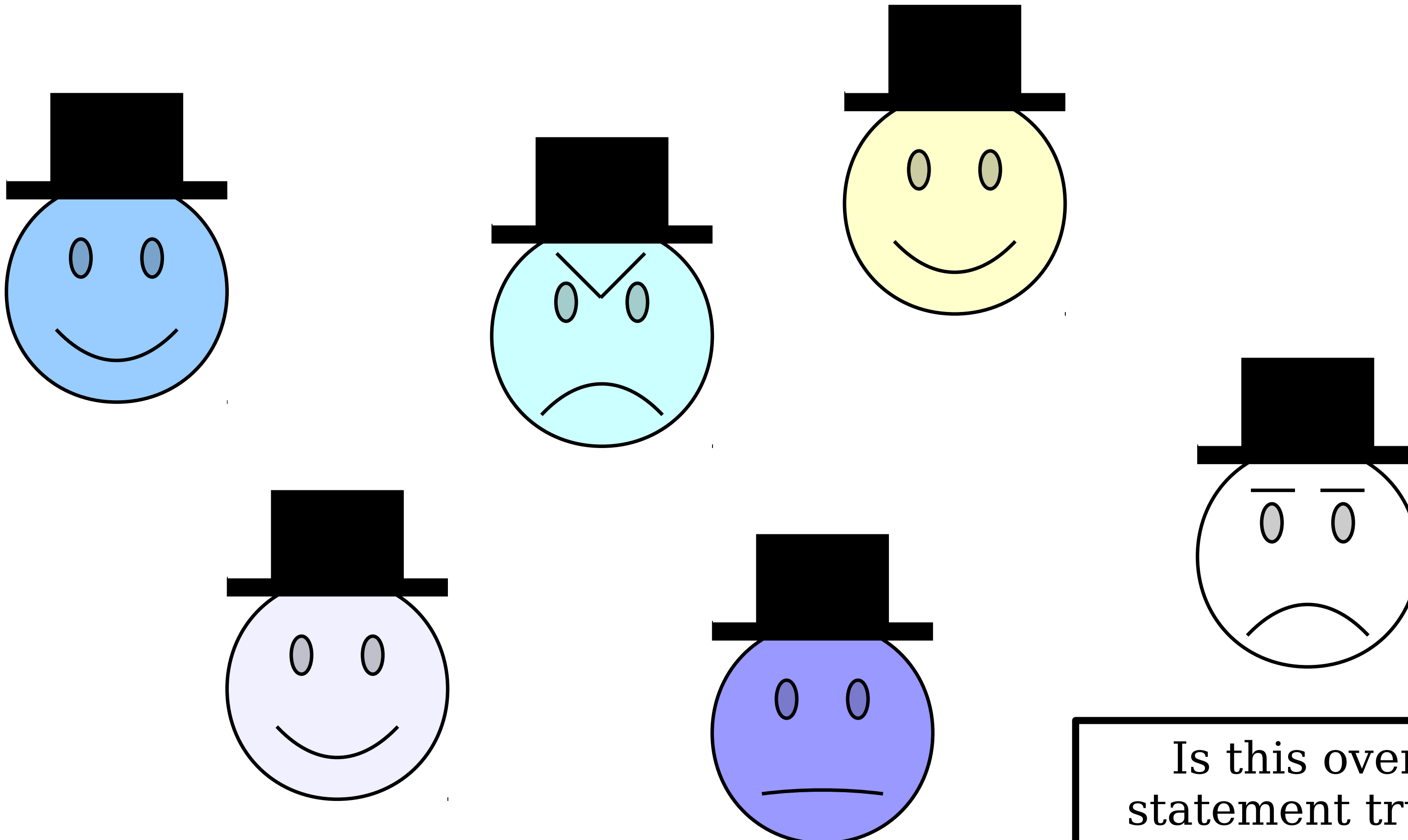
The Universal Quantifier



Is this part of the statement true or false?

~~$(\forall x.\ Smiling(x))$~~ $\to$ $(\forall y.\ WearingHat(y))$

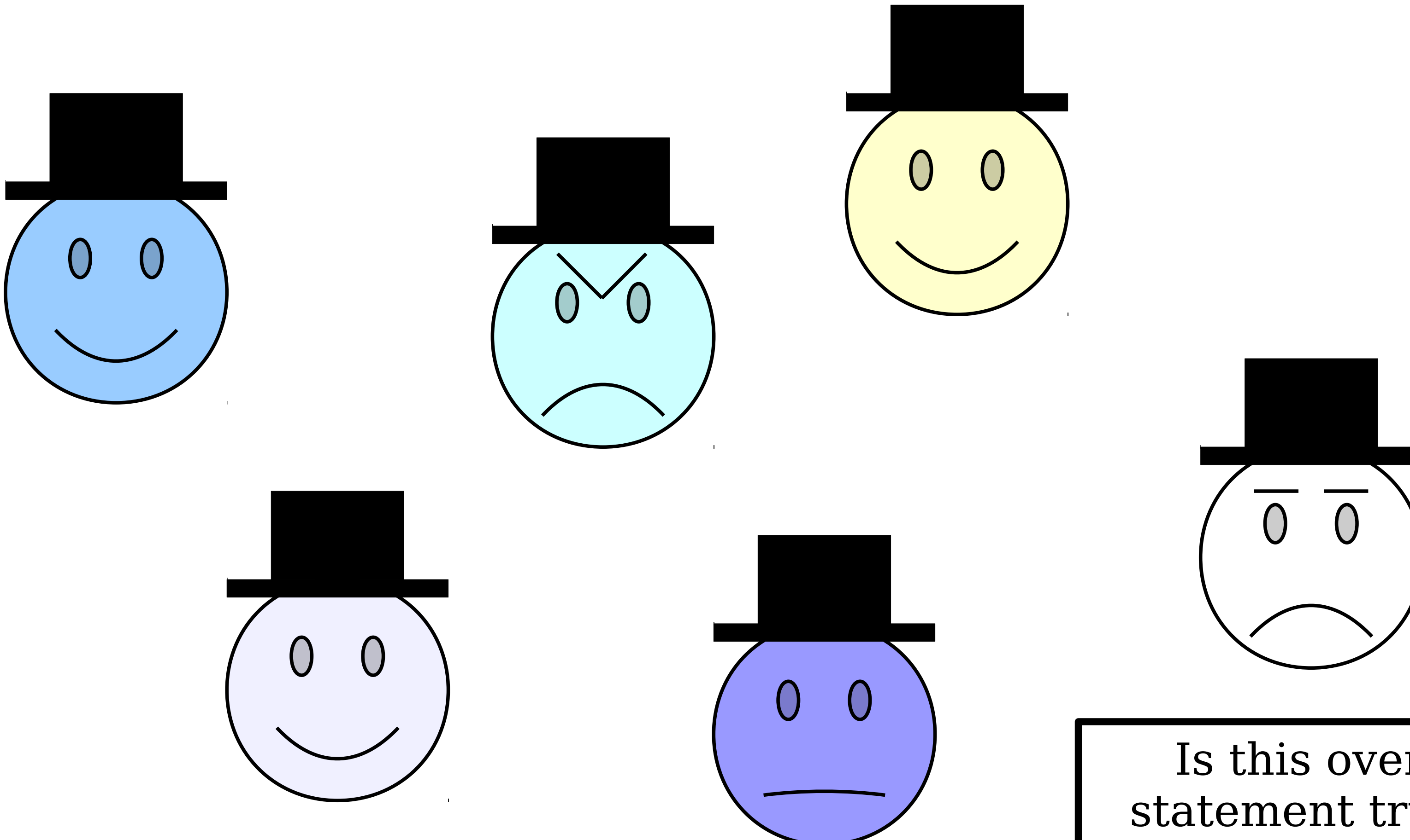
The Universal Quantifier



Is this overall statement true or false in this scenario?

~~$(\forall x.\ \mathit{Smiling}(x))$~~ $\rightarrow (\forall y.\ \mathit{WearingHat}(y))$

The Universal Quantifier



Is this overall statement true or false in this scenario?

$$(\forall x.\ Smiling(x)) \rightarrow (\forall y.\ WearingHat(y))$$

Fun with Edge Cases

$$\forall x.\ Smiling(x)$$

Fun with Edge Cases

Universally-quantified statements are said to be ***vacuously true*** in empty worlds.

$$\forall x.\ Smiling(x)$$

Let’s take a quick break!

Translating into First-Order Logic

Translating Into Logic

- First-order logic is an excellent tool for manipulating definitions and theorems to learn more about them.
- Need to take a negation? Translate your statement into FOL, negate it, then translate it back.
- Want to prove something by contrapositive? Translate your implication into FOL, take the contrapositive, then translate it back.

Translating Into Logic

- When translating from English into first-order logic, we recommend that you

  think of first-order logic as a mathematical programming language.

- Your goal is to learn how to combine basic concepts (quantifiers, connectives, etc.) together in ways that say what you mean.

Using the predicates

- *Smiling*(x), which states that x is smiling, and
- *WearingHat*(x), which states that x is wearing a hat,

write a sentence in first-order logic that says

some smiling person wears a hat.

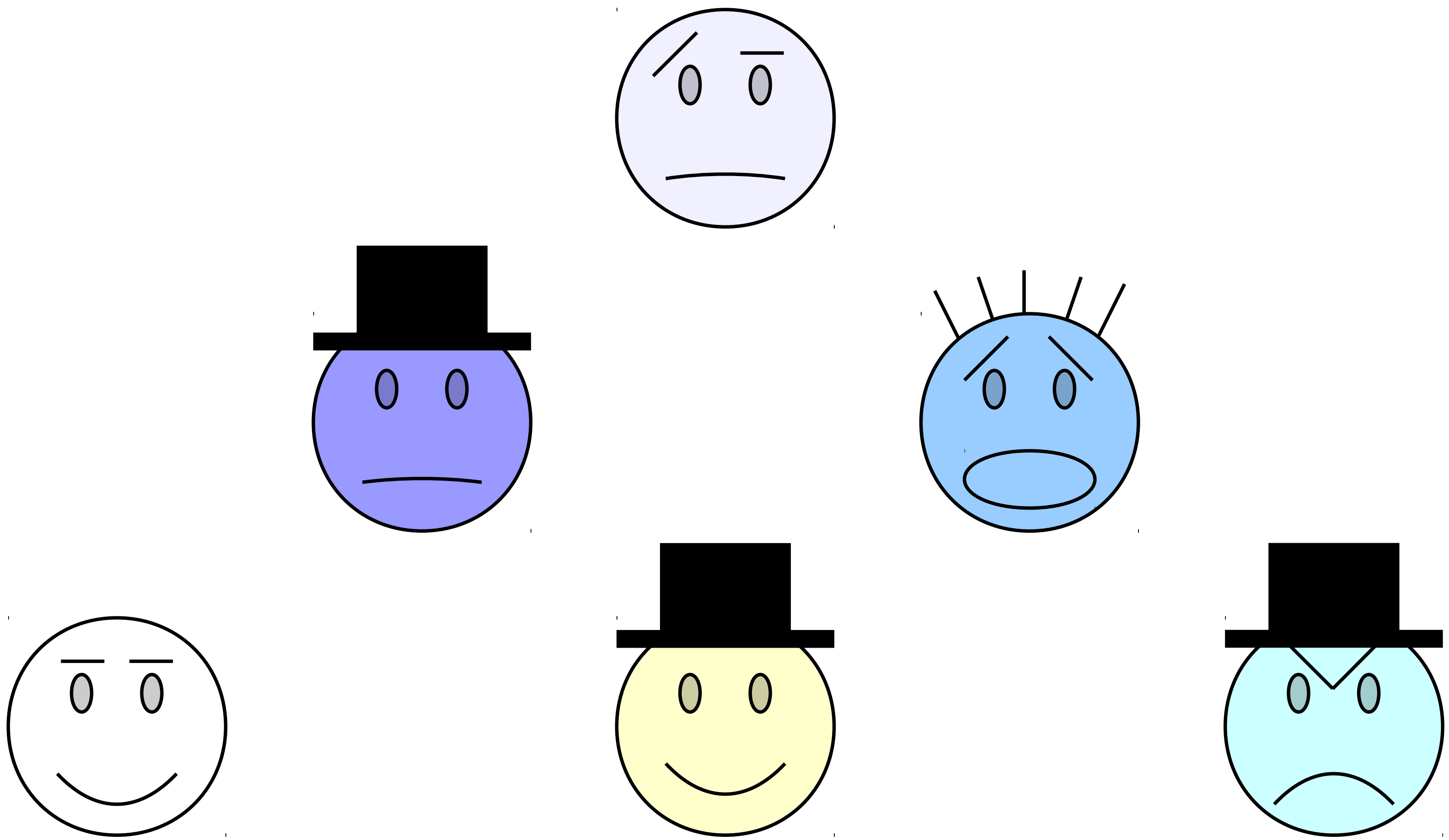
Try it yourself: Give this your best shot – it's okay if you're not sure!

Respond at pollev.com/zhenglian740

"Some smiling person wears a hat."

$$\exists x.\ (Smiling(x) \land WearingHat(x))$$

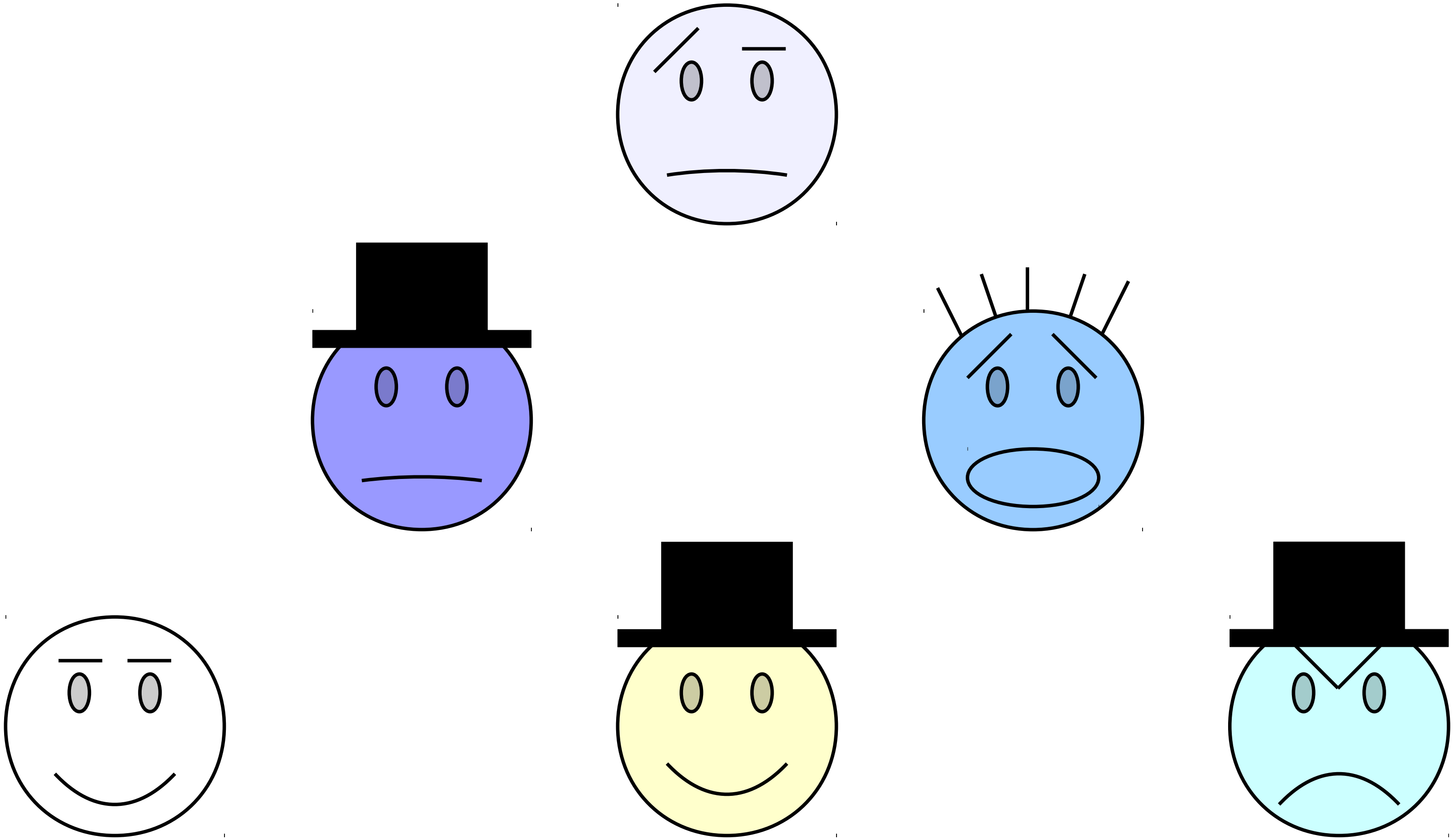
$$\exists x.\ (Smiling(x) \to WearingHat(x))$$



"Some smiling person wears a hat."

$\exists x. (Smiling(x) \land WearingHat(x))$

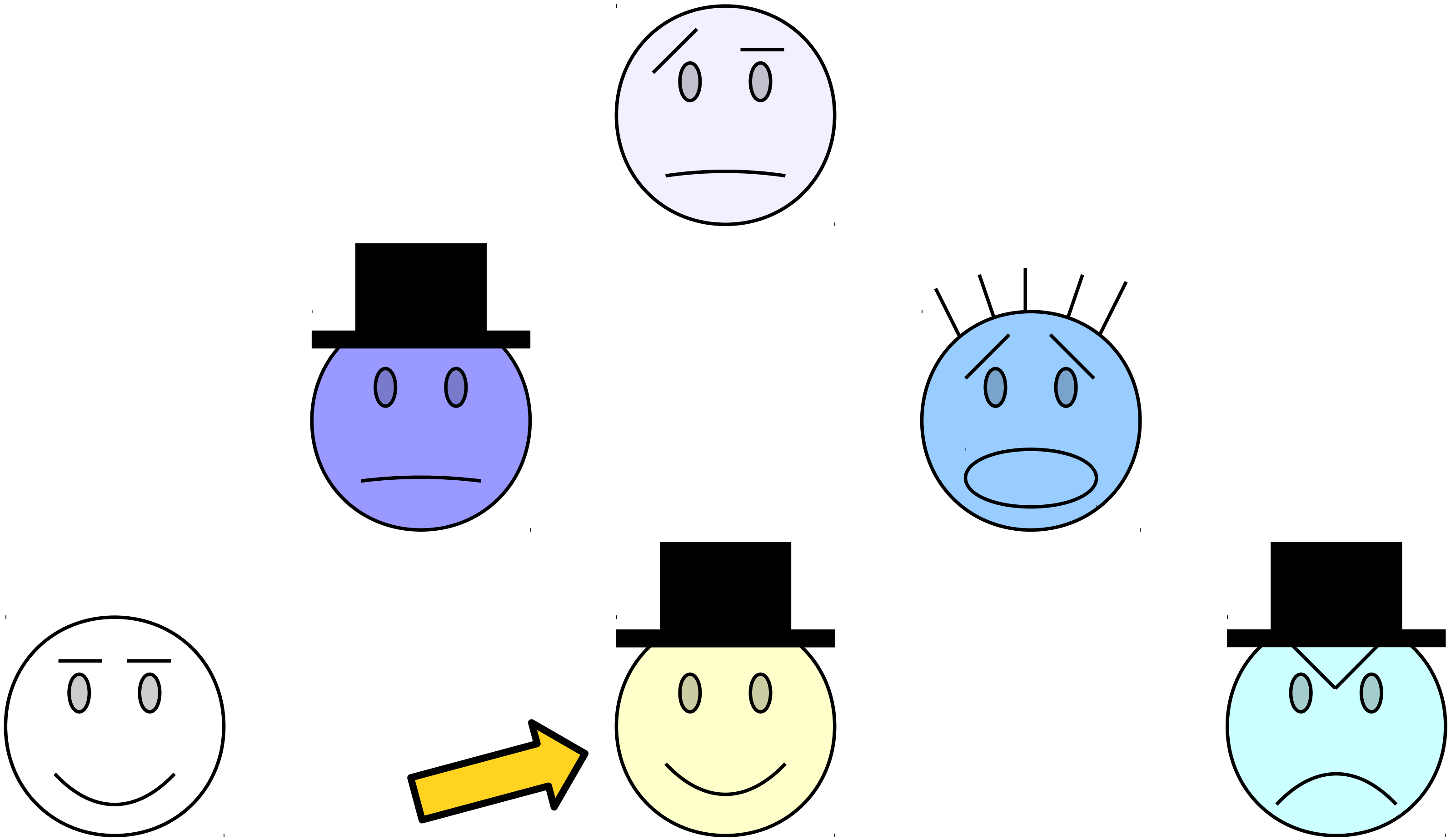
$\exists x. (Smiling(x) \to WearingHat(x))$



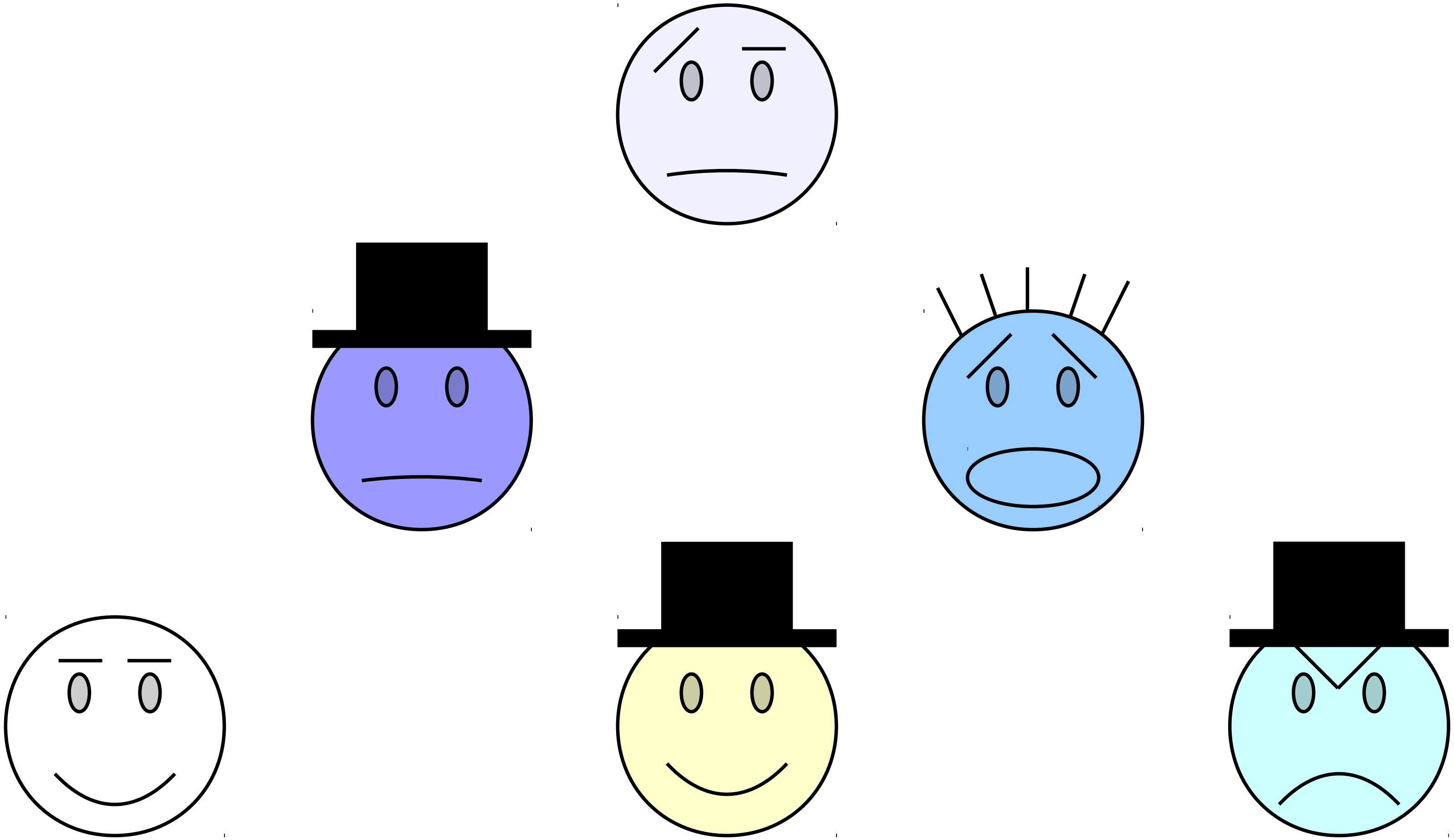
"Some smiling person wears a hat."

$\exists x.\ (Smiling(x) \land WearingHat(x))$

$\exists x.\ (Smiling(x) \to WearingHat(x))$



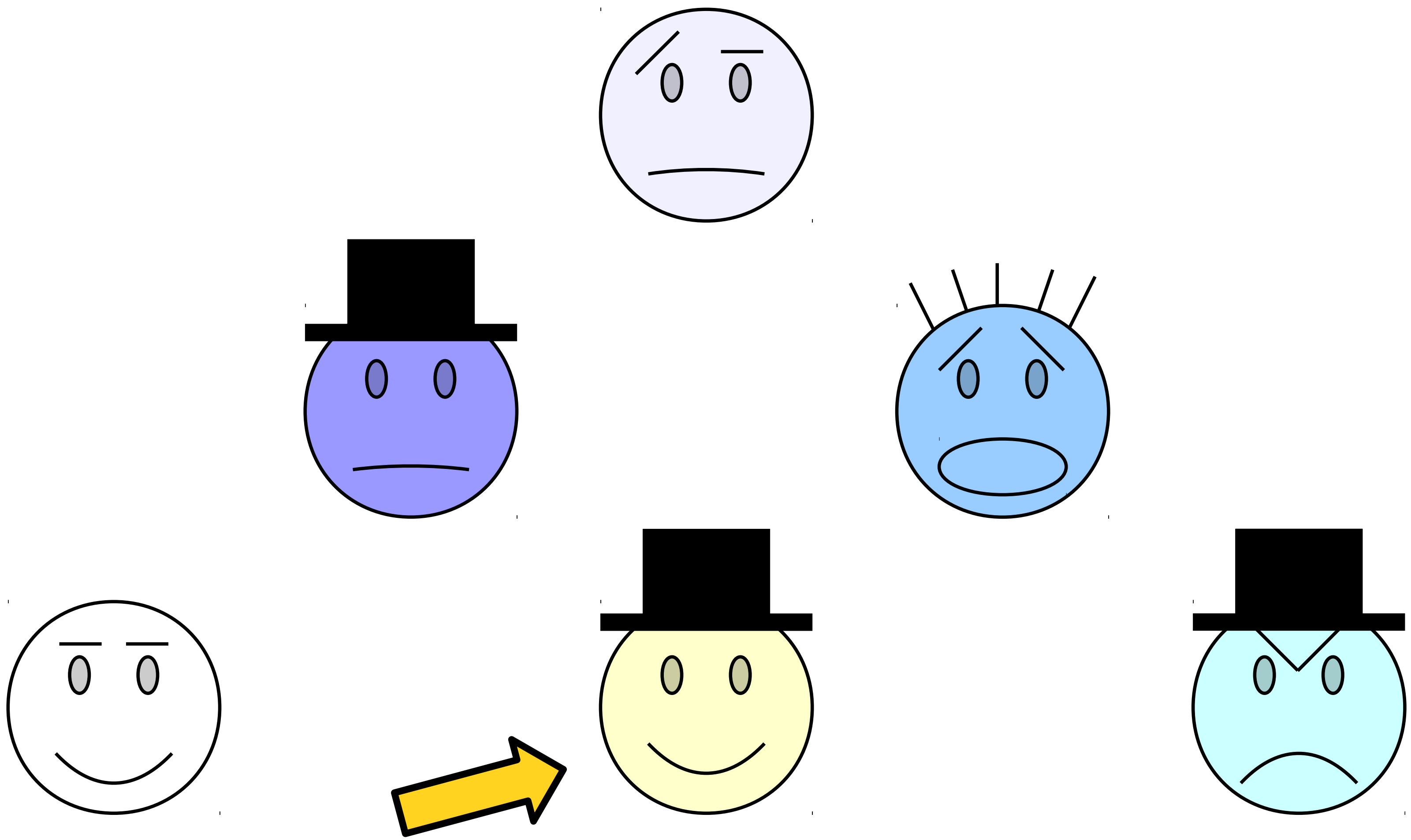
"Some smiling person wears a hat."	***True***
$\exists x.\ (Smiling(x) \land WearingHat(x))$	
$\exists x.\ (Smiling(x) \to WearingHat(x))$	



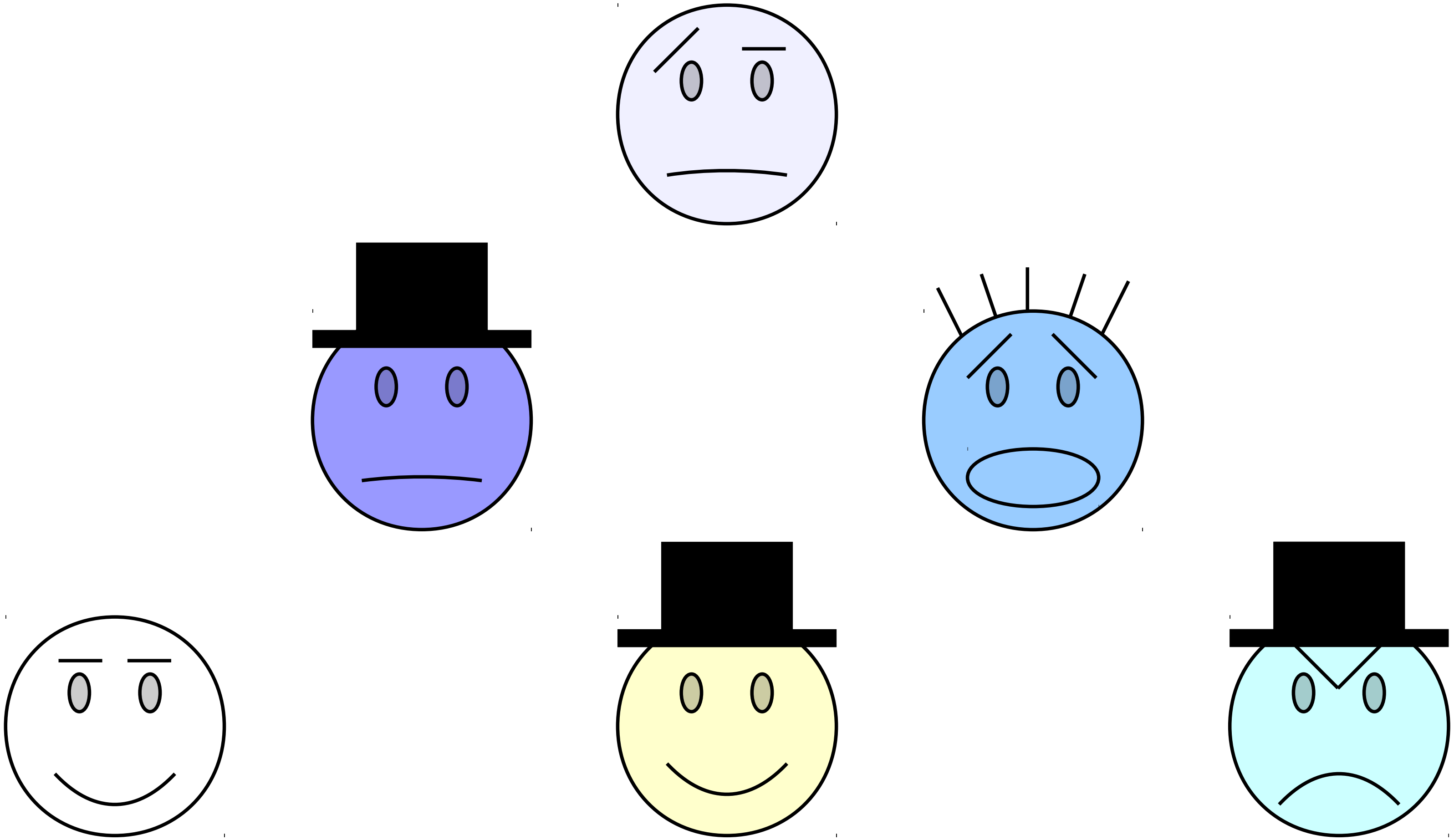
“Some smiling person wears a hat.” ***True***

$\exists x. \, (Smiling(x) \land WearingHat(x))$

$\exists x. \, (Smiling(x) \to WearingHat(x))$



"Some smiling person wears a hat."	***True***
$\exists x. \, (Smiling(x) \land WearingHat(x))$	***True***
$\exists x. \, (Smiling(x) \to WearingHat(x))$	



"Some smiling person wears a hat."	***True***
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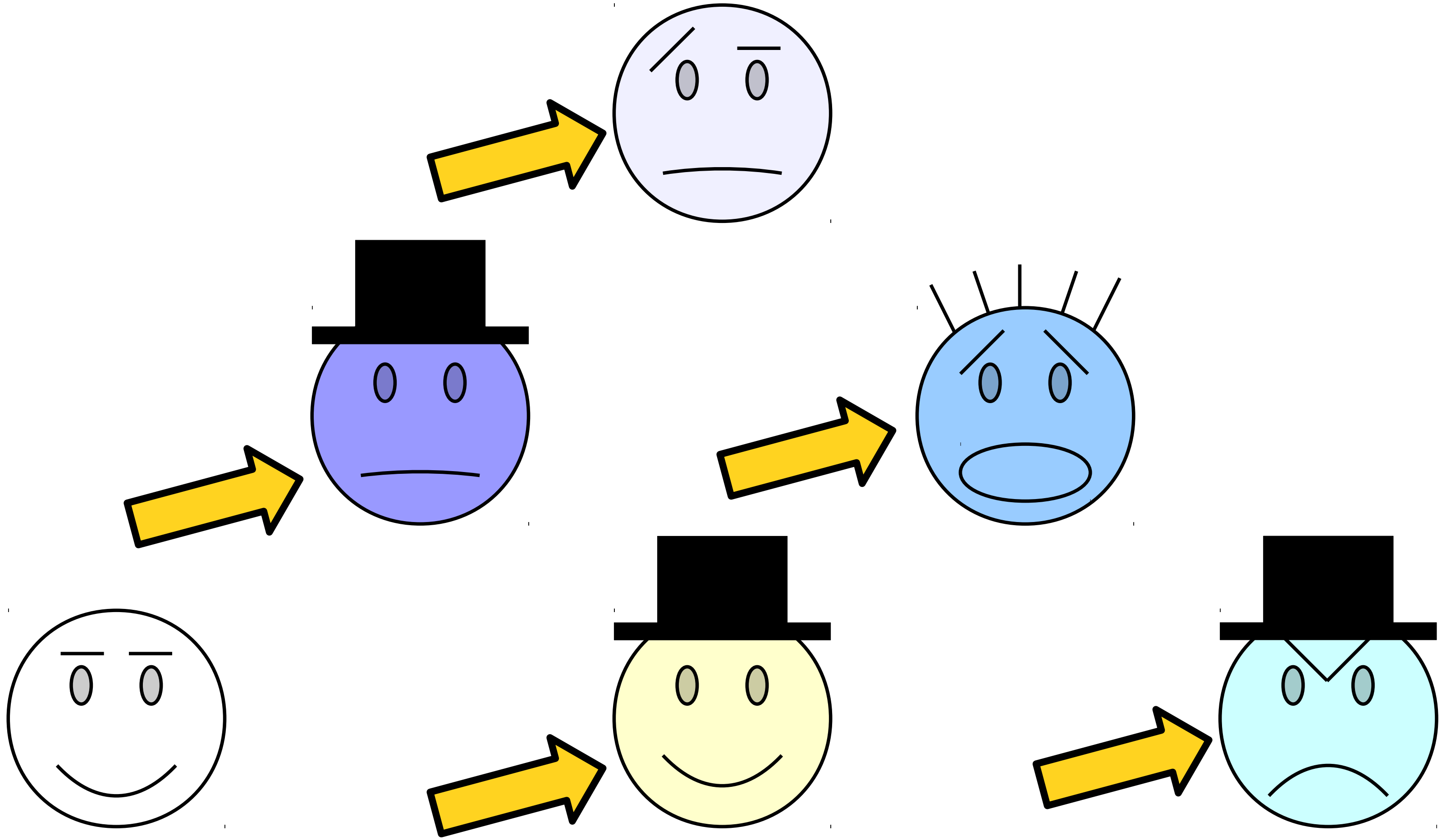
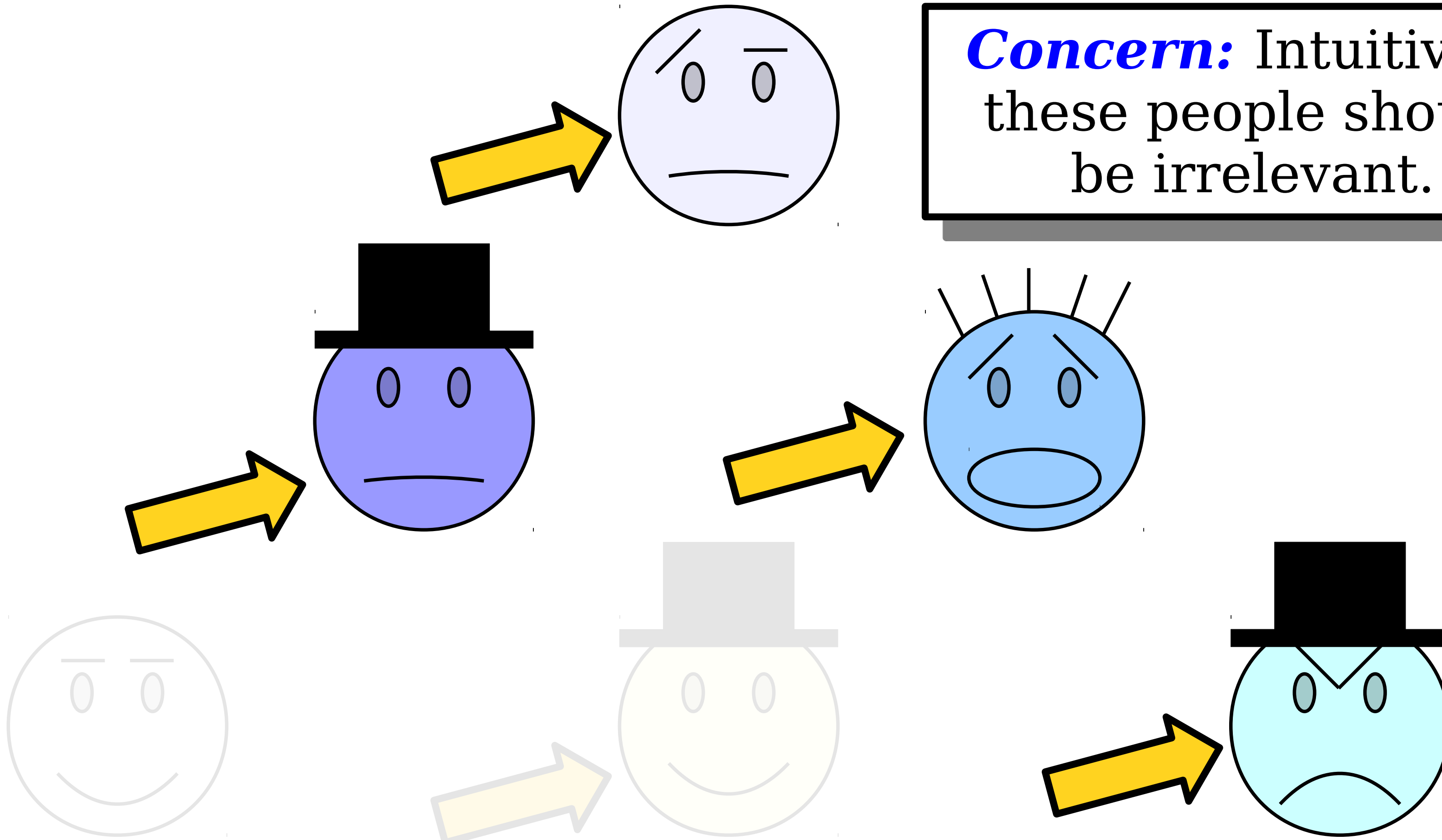


"Some smiling person wears a hat."	***True***
$\exists x.\ (\mathit{Smiling}(x) \land \mathit{WearingHat}(x))$	***True***
$\exists x.\ (\mathit{Smiling}(x) \to \mathit{WearingHat}(x))$	***True***



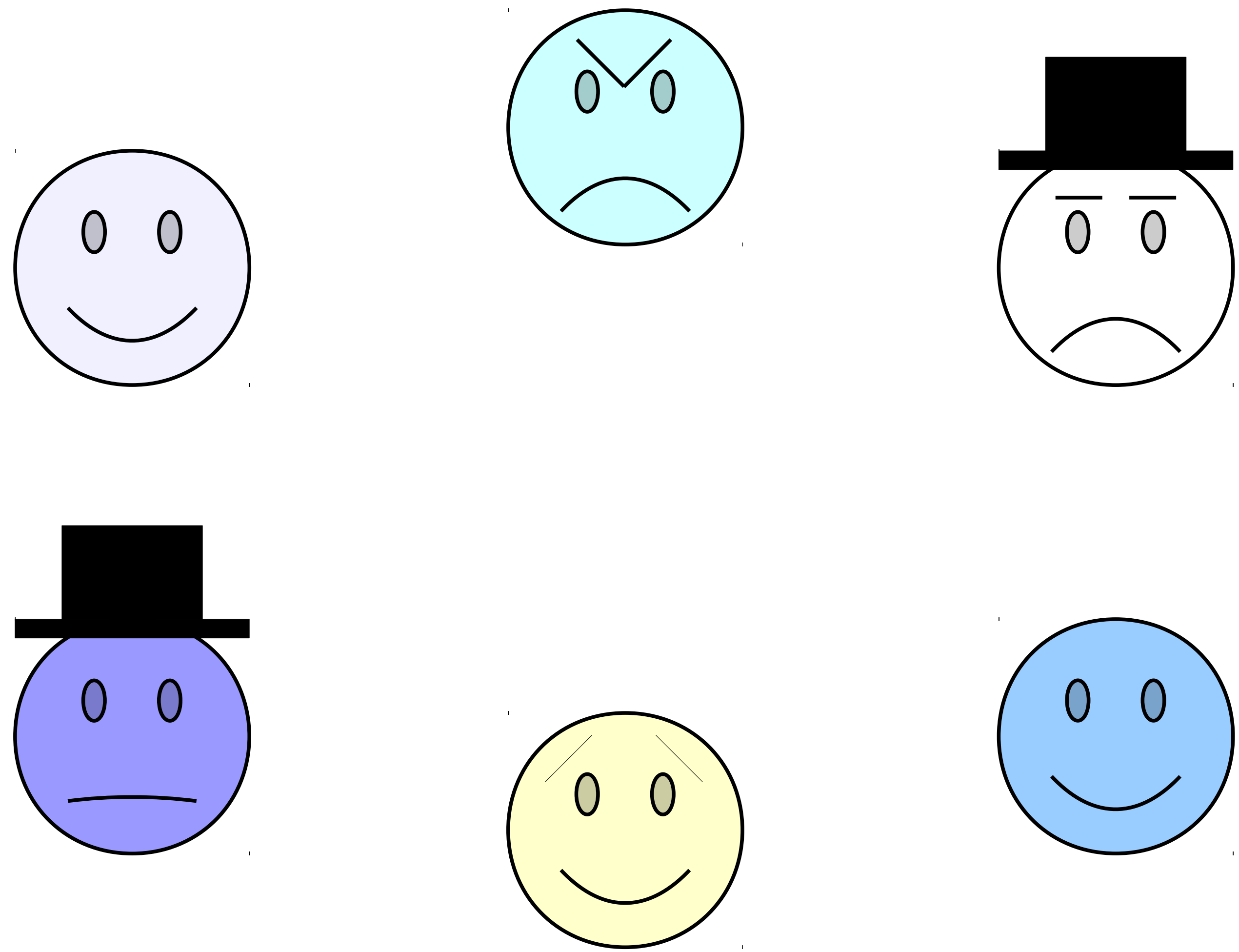
Concern: Intuitively, these people should be irrelevant.

"Some smiling person wears a hat."	***True***
$\exists x.\ (Smiling(x) \land WearingHat(x))$	***True***
$\exists x.\ (Smiling(x) \to WearingHat(x))$	***True***

"Some smiling person wears a hat."

$$\exists x.\ (Smiling(x) \land WearingHat(x))$$

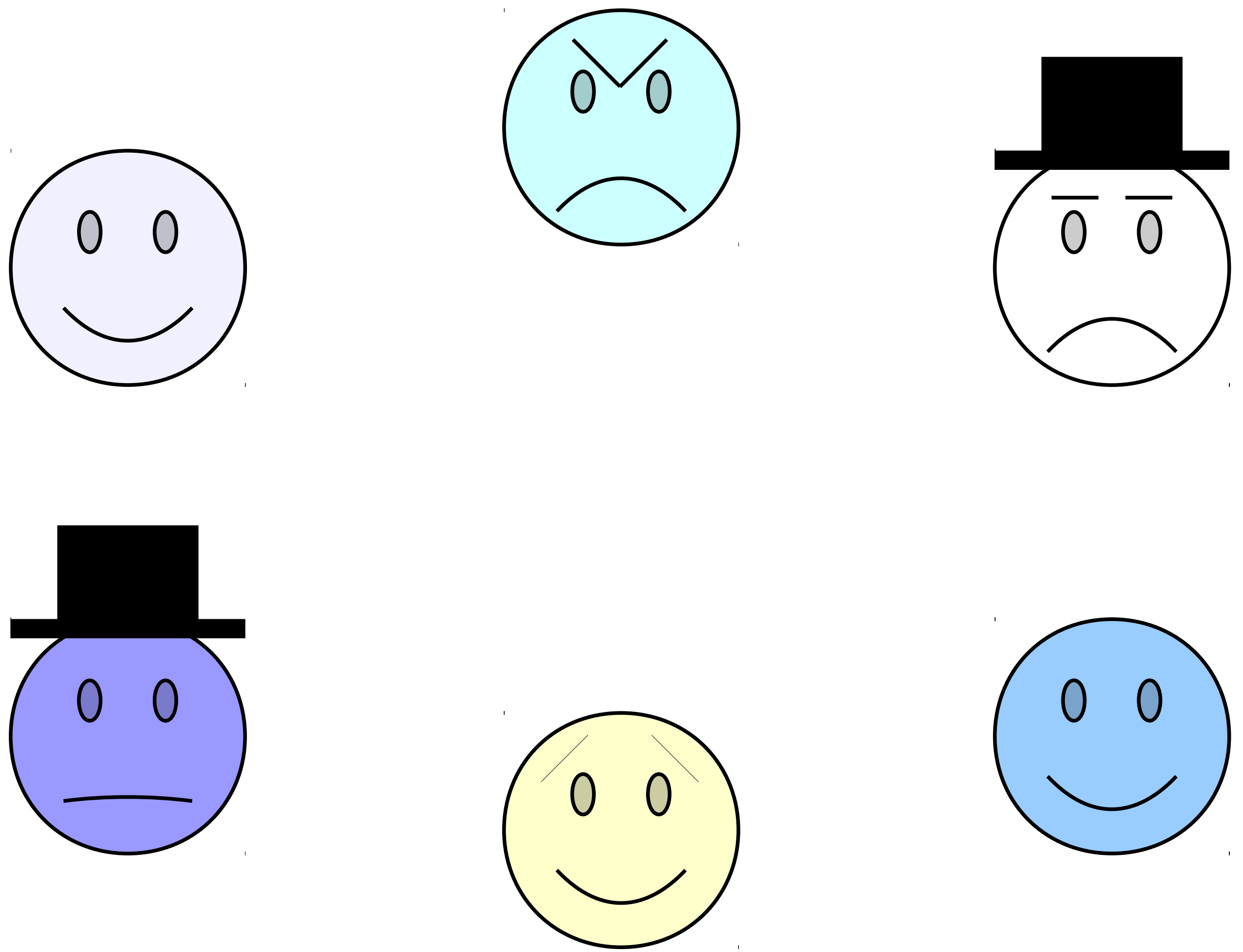
$$\exists x.\ (Smiling(x) \to WearingHat(x))$$



"Some smiling person wears a hat."

$\exists x. (Smiling(x) \land WearingHat(x))$

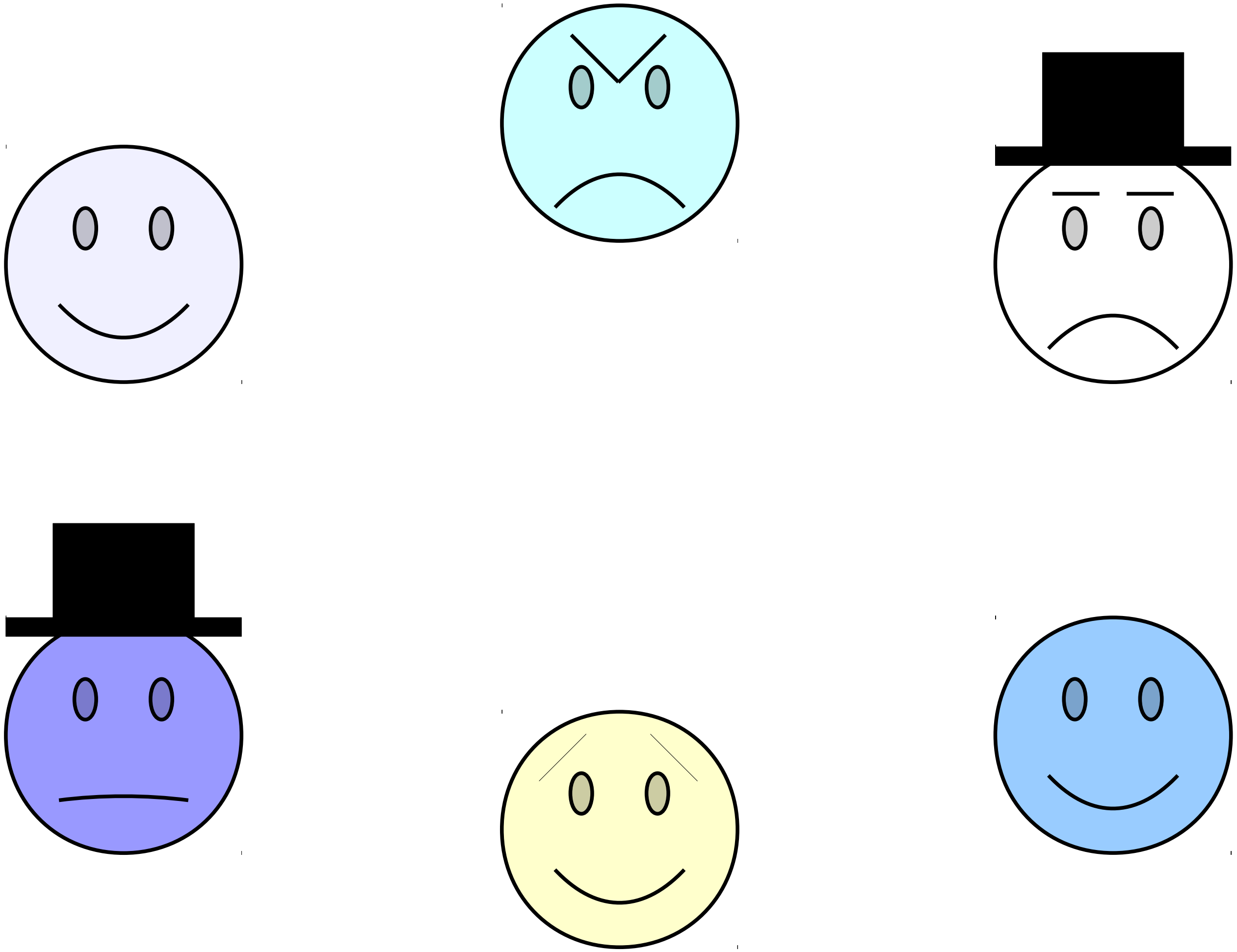
$\exists x. (Smiling(x) \rightarrow WearingHat(x))$



"Some smiling person wears a hat."

$\exists x.\ (Smiling(x) \land WearingHat(x))$

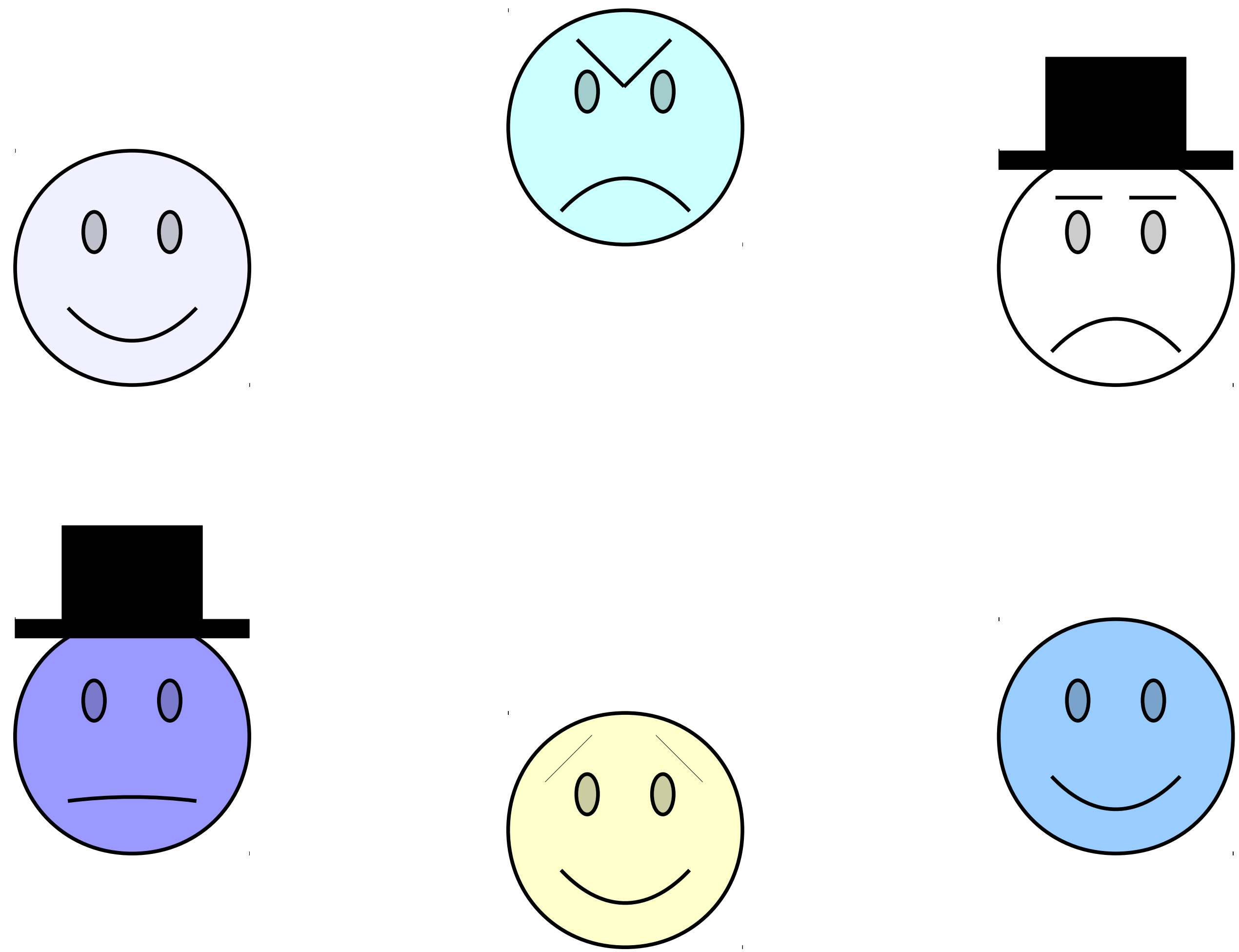
$\exists x.\ (Smiling(x) \to WearingHat(x))$



"Some smiling person wears a hat." ***False***

$\exists x.\ (Smiling(x) \land WearingHat(x))$

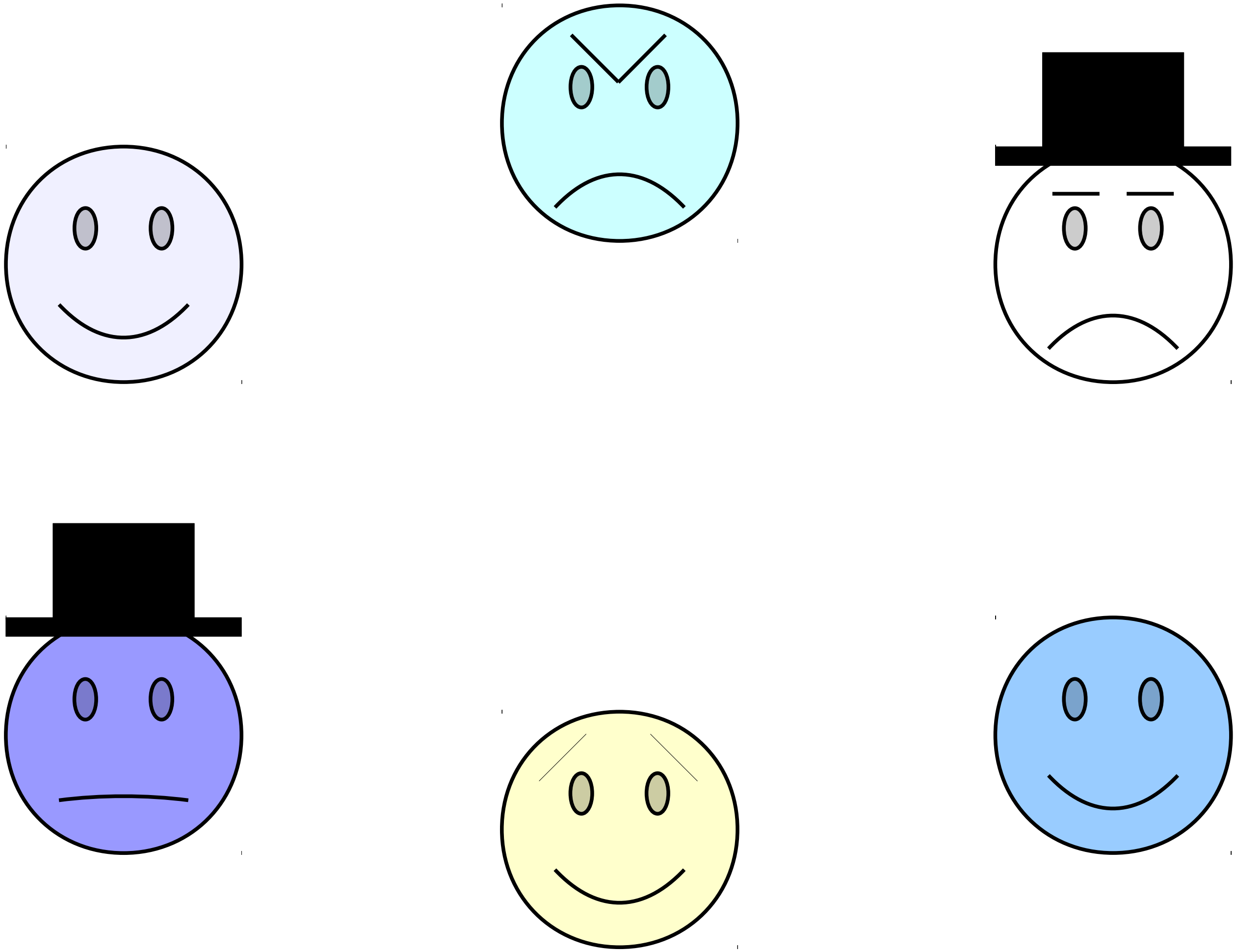
$\exists x.\ (Smiling(x) \to WearingHat(x))$



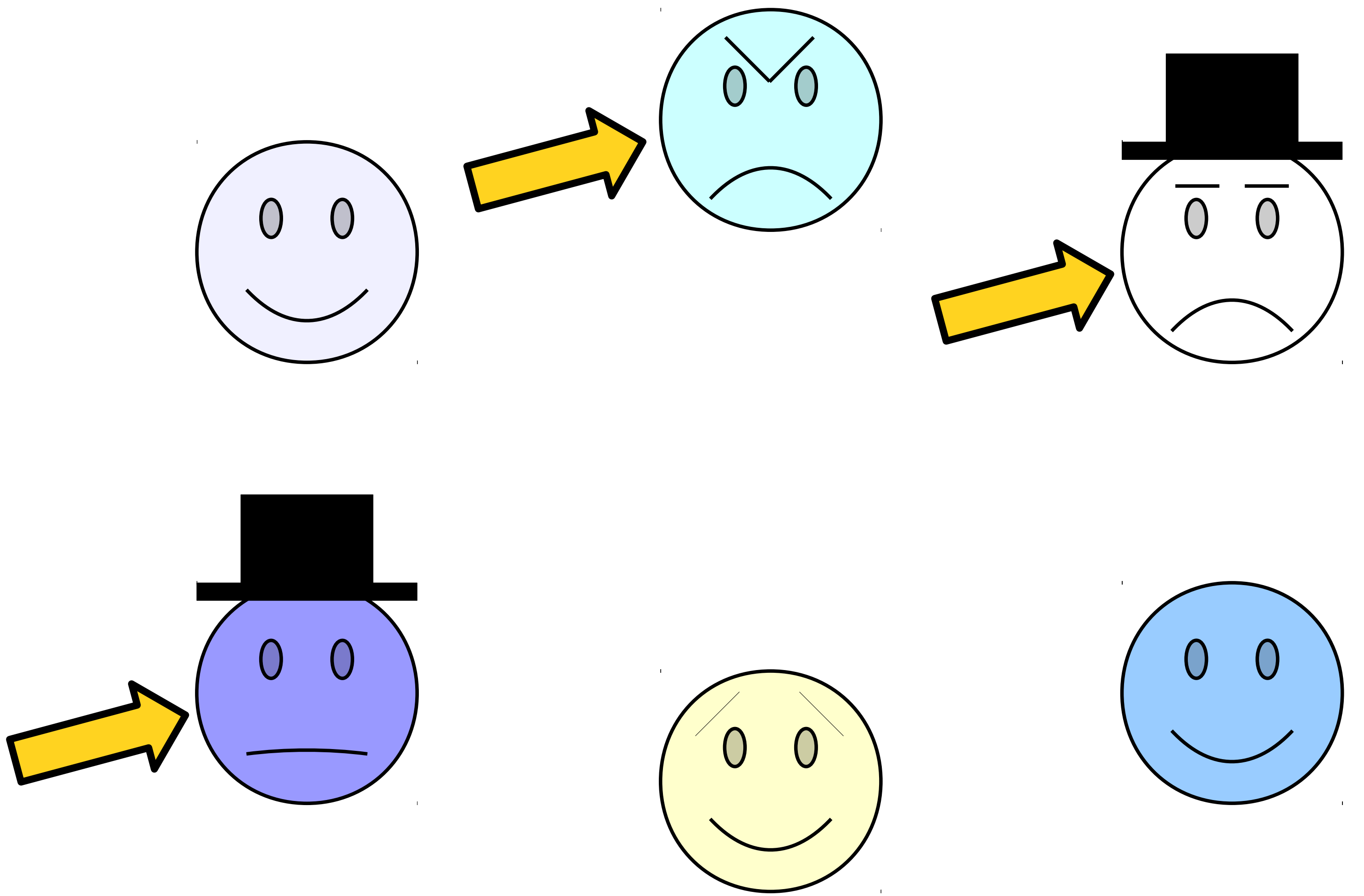
"Some smiling person wears a hat." ***False***

$\exists x.\ (Smiling(x) \land WearingHat(x))$

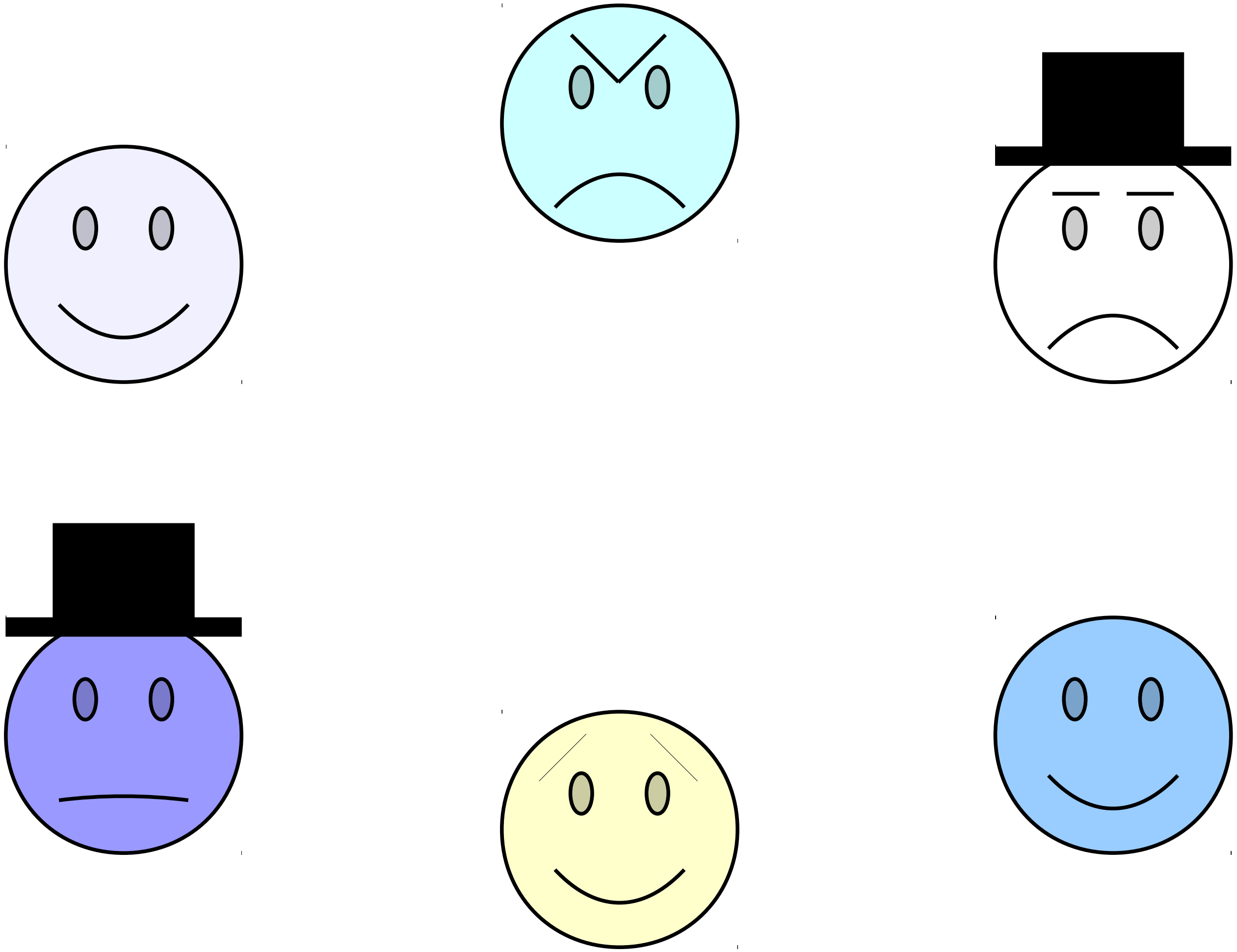
$\exists x.\ (Smiling(x) \to WearingHat(x))$



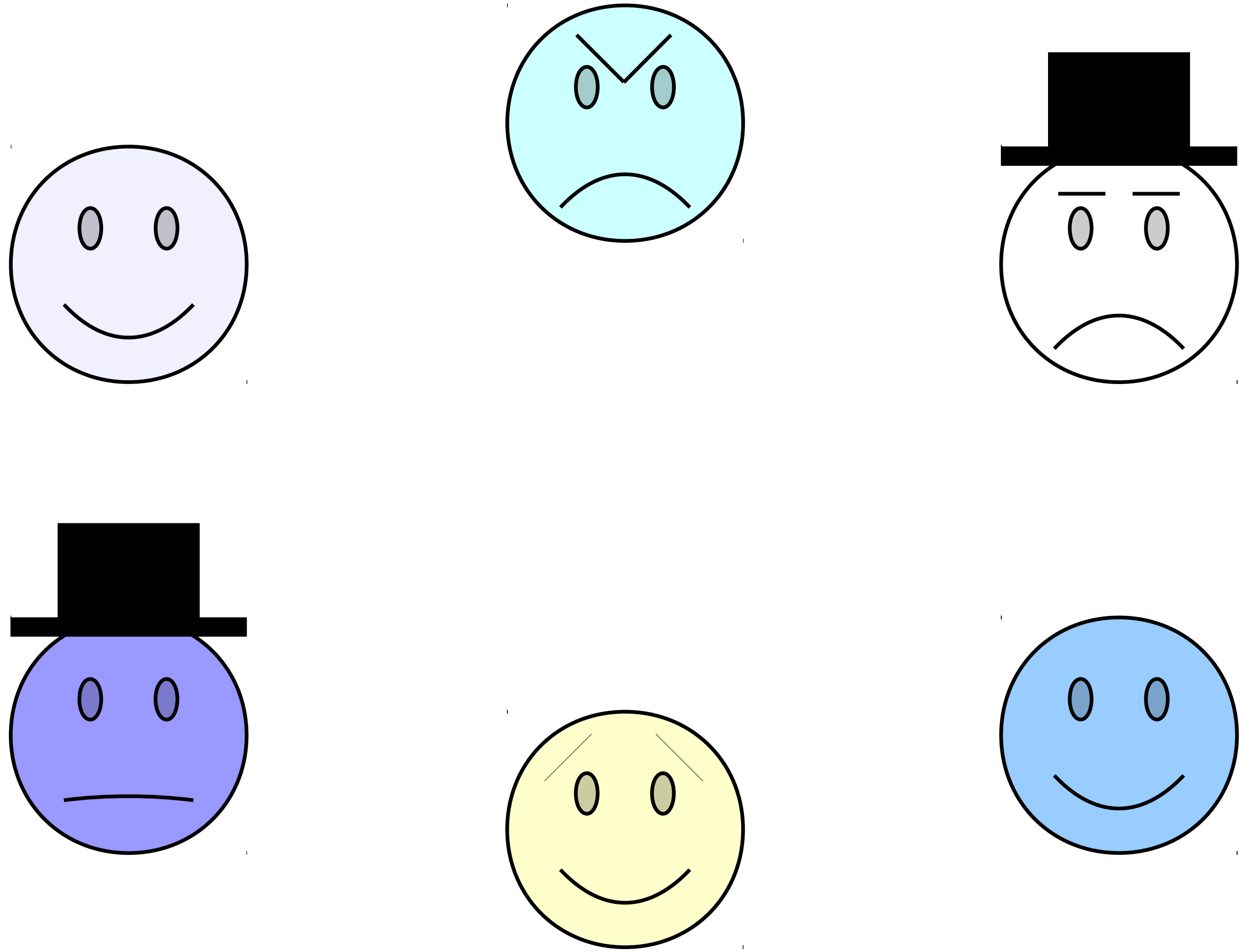
"Some smiling person wears a hat."	***False***
$\exists x.\ (Smiling(x) \land WearingHat(x))$	***False***
$\exists x.\ (Smiling(x) \to WearingHat(x))$	



"Some smiling person wears a hat."	***False***
$\exists x.\ (Smiling(x) \land WearingHat(x))$	***False***
$\exists x.\ (Smiling(x) \to WearingHat(x))$	***True***



"Some smiling person wears a hat."	***False***
$\exists x. \, (Smiling(x) \land WearingHat(x))$	***False***
$\exists x. \, (Smiling(x) \to WearingHat(x))$	***True***



"Some smiling person wears a hat."	***False***
$\exists x.\ (Smiling(x) \land WearingHat(x))$	***False***
~~$\exists x.\ (Smiling(x) \to WearingHat(x))$~~	***True***

“Some *P* is a *Q*”

translates as

$$\exists x.\ (P(x) \land Q(x))$$

Useful Intuition:

Existentially-quantified statements are false unless there's a positive example.

$$\exists x.\ (P(x) \land Q(x))$$

If *x* is an example, it *must* have property *P* on top of property *Q*.

Using the predicates

- *Smiling*(x), which states that x is smiling, and
- *WearingHat*(x), which states that x is wearing a hat,

write a sentence in first-order logic that says

every smiling person wears a hat.

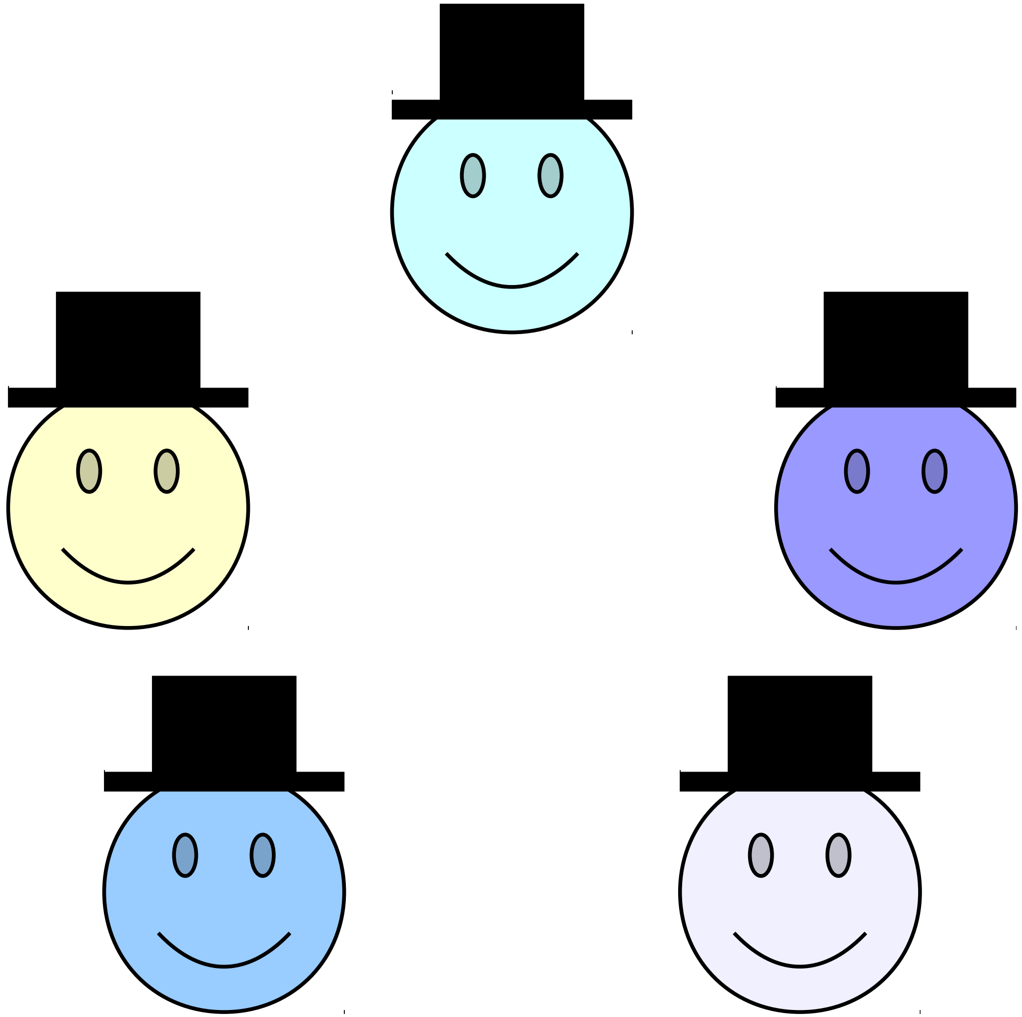
Try it yourself: Give this your best shot – it's okay if you're not sure!

Respond at pollev.com/zhenglian740

"Every smiling person wears a hat."

$$\forall x.\ (Smiling(x) \land WearingHat(x))$$

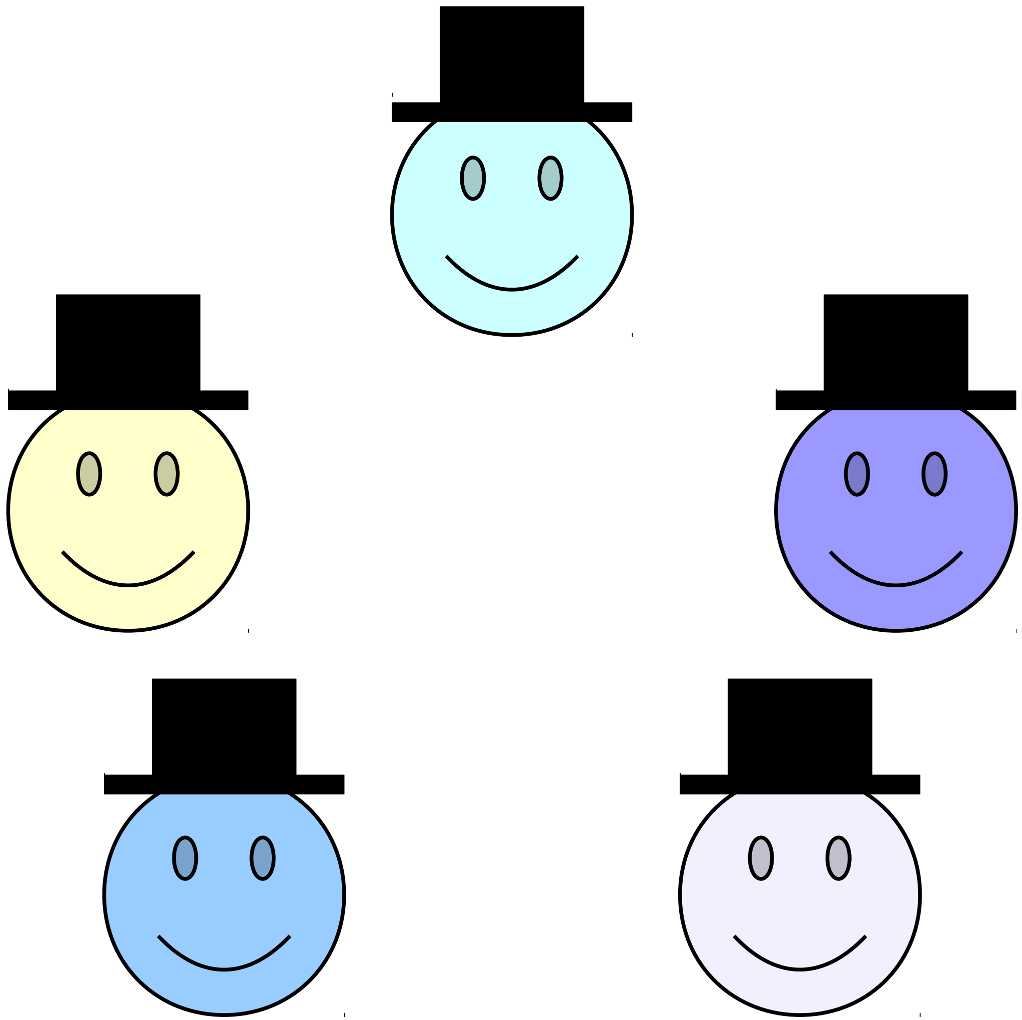
$$\forall x.\ (Smiling(x) \to WearingHat(x))$$



“Every smiling person wears a hat.”

$$\forall x.\ (Smiling(x) \land WearingHat(x))$$

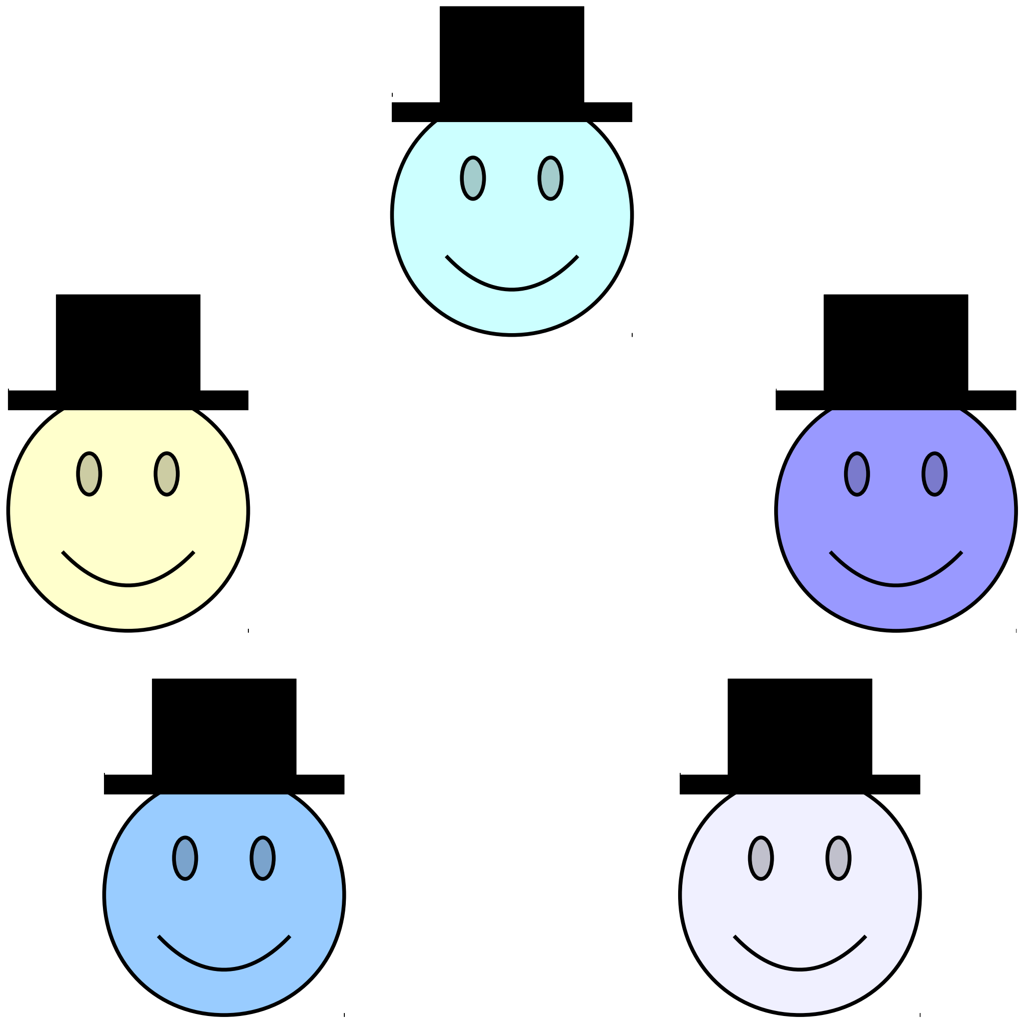
$$\forall x.\ (Smiling(x) \to WearingHat(x))$$



“Every smiling person wears a hat.” ***True***

$\forall x. \, (Smiling(x) \land WearingHat(x))$

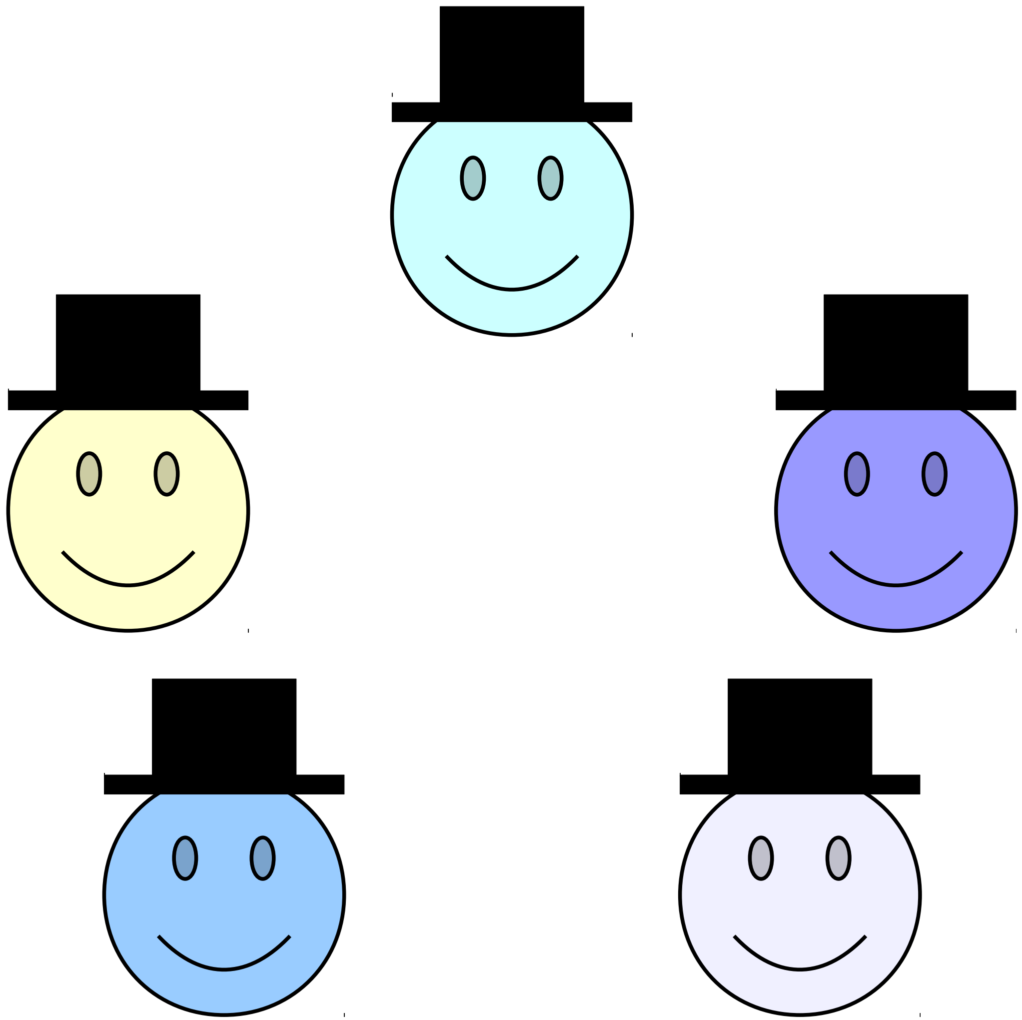
$\forall x. \, (Smiling(x) \to WearingHat(x))$



"Every smiling person wears a hat." ***True***

$\forall x. (Smiling(x) \land WearingHat(x))$ ***True***

$\forall x. (Smiling(x) \rightarrow WearingHat(x))$



"Every smiling person wears a hat."	***True***
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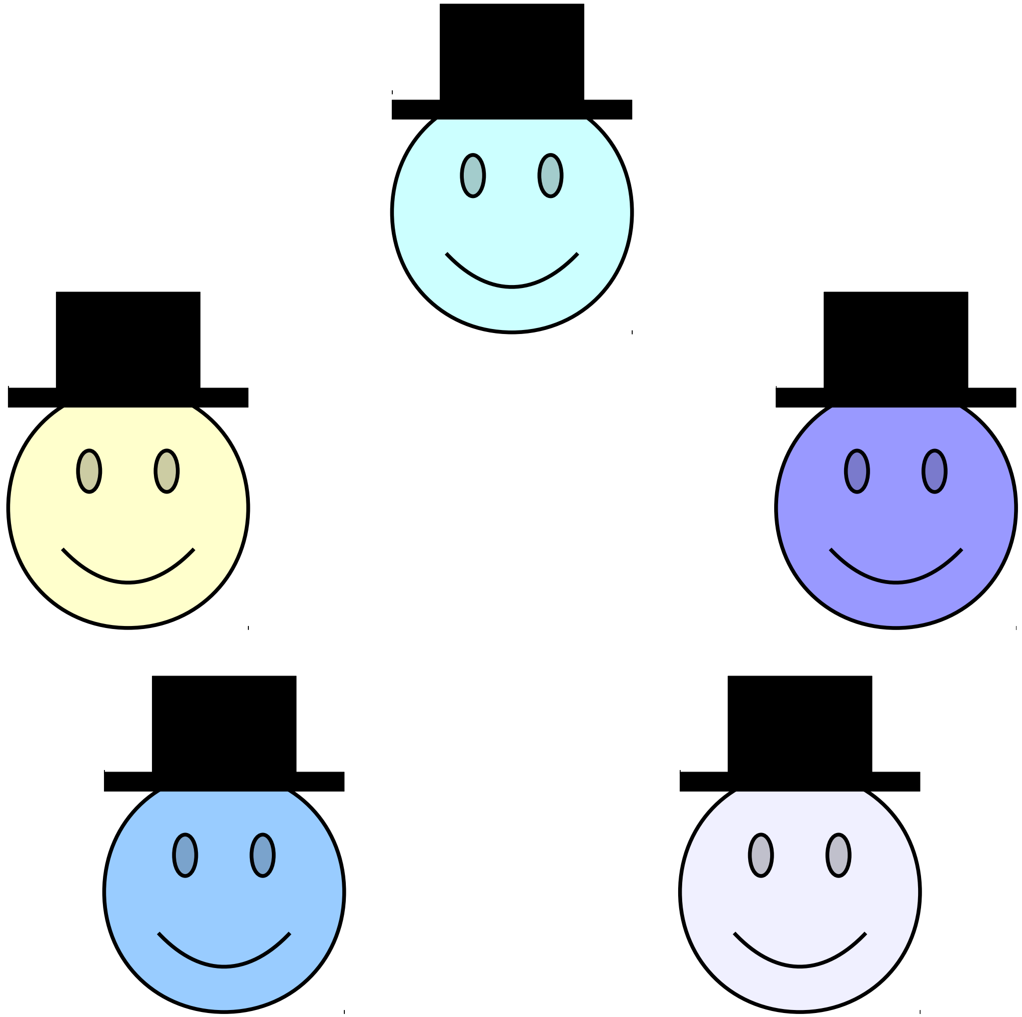

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"Every smiling person wears a hat."

$$\forall x.\ (Smiling(x) \land WearingHat(x))$$

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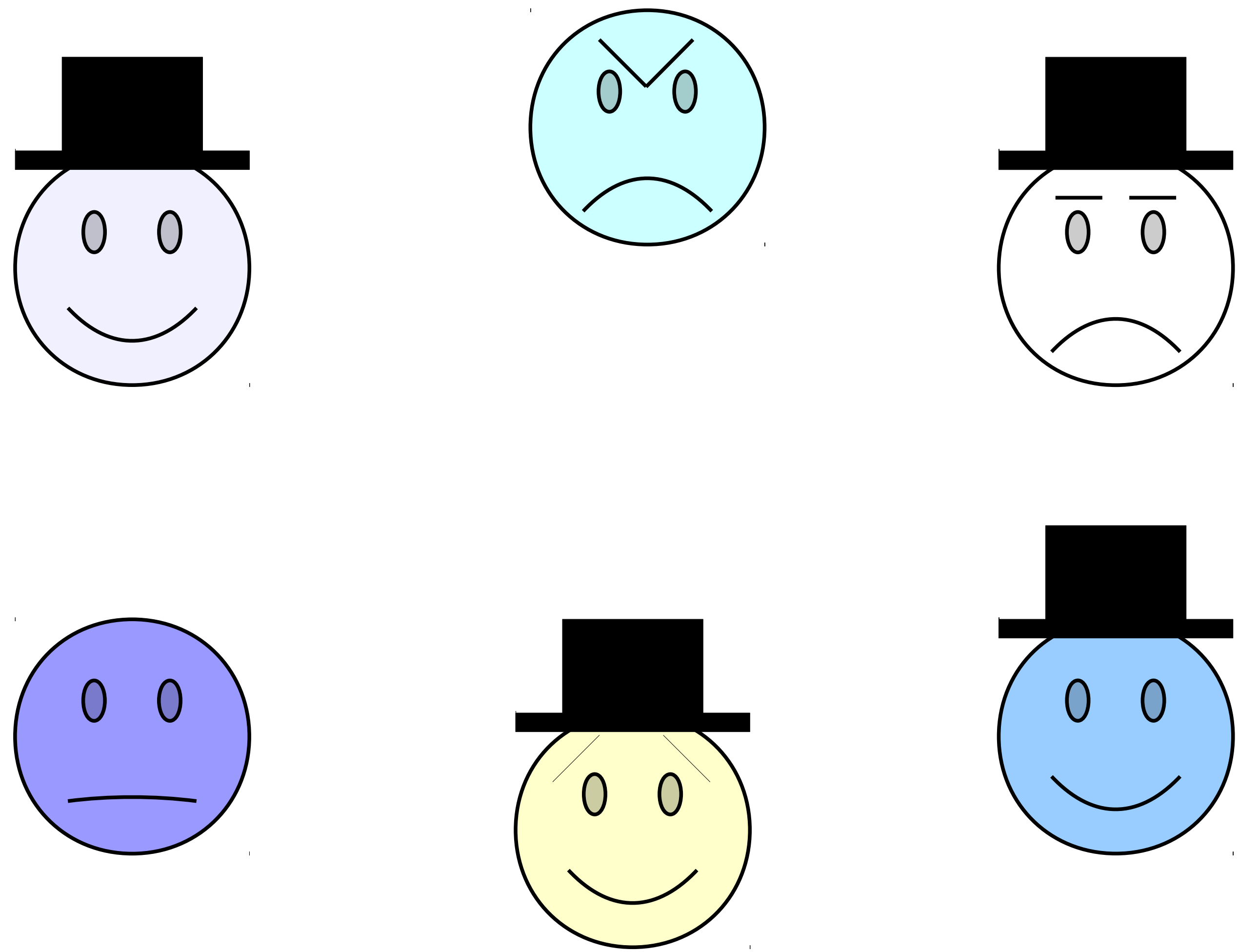

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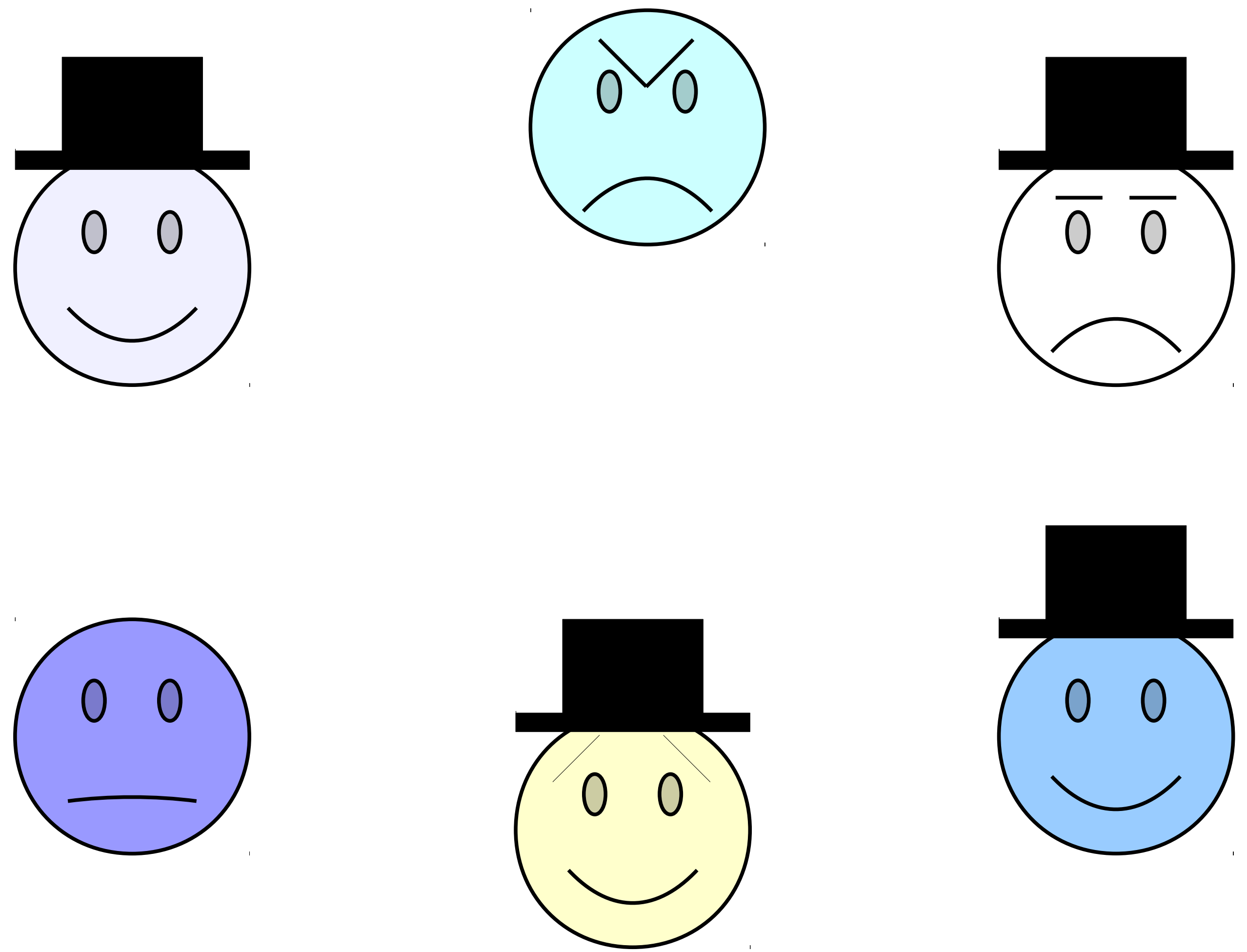

"Every smiling person wears a hat." ***True***

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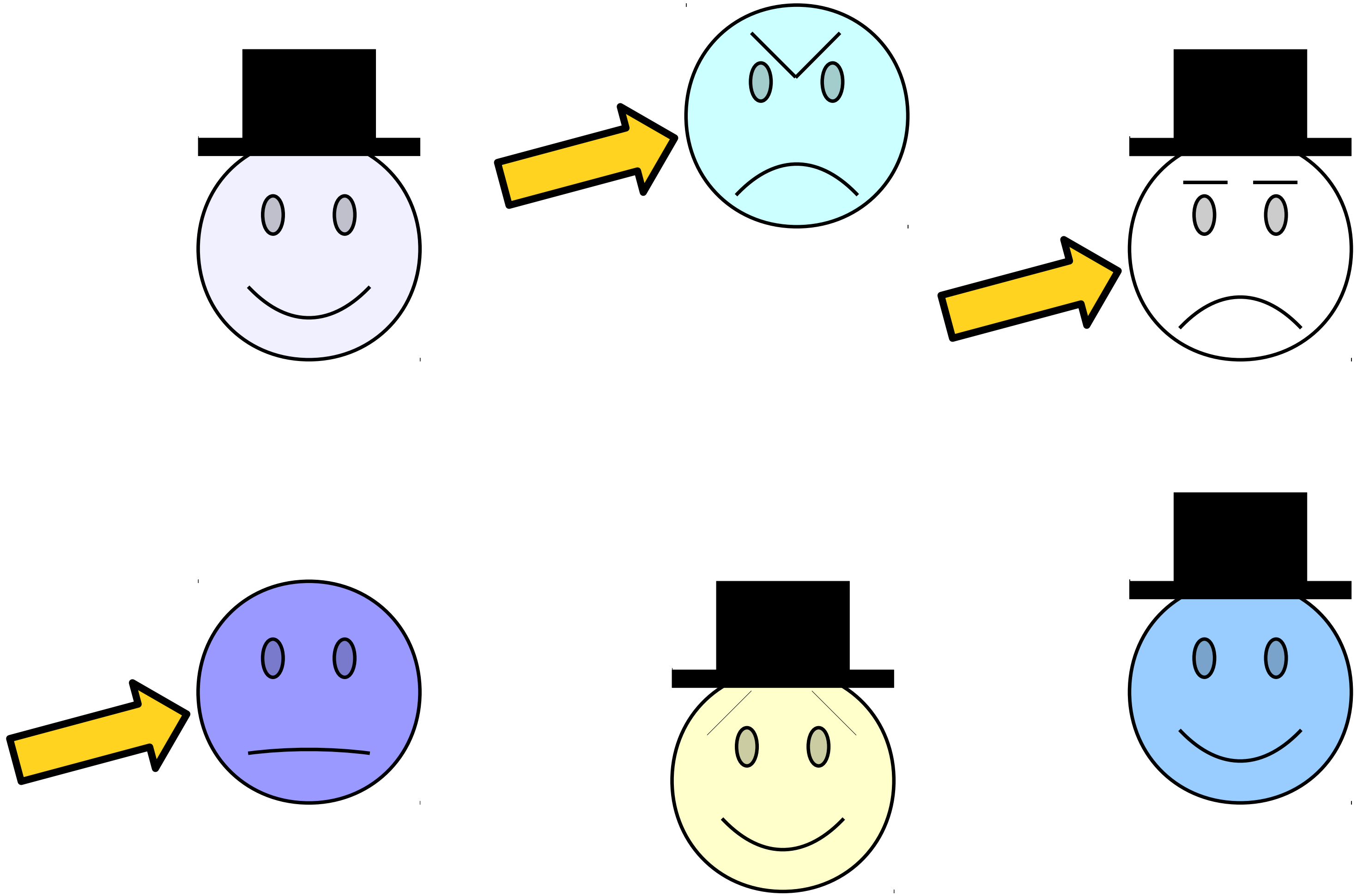

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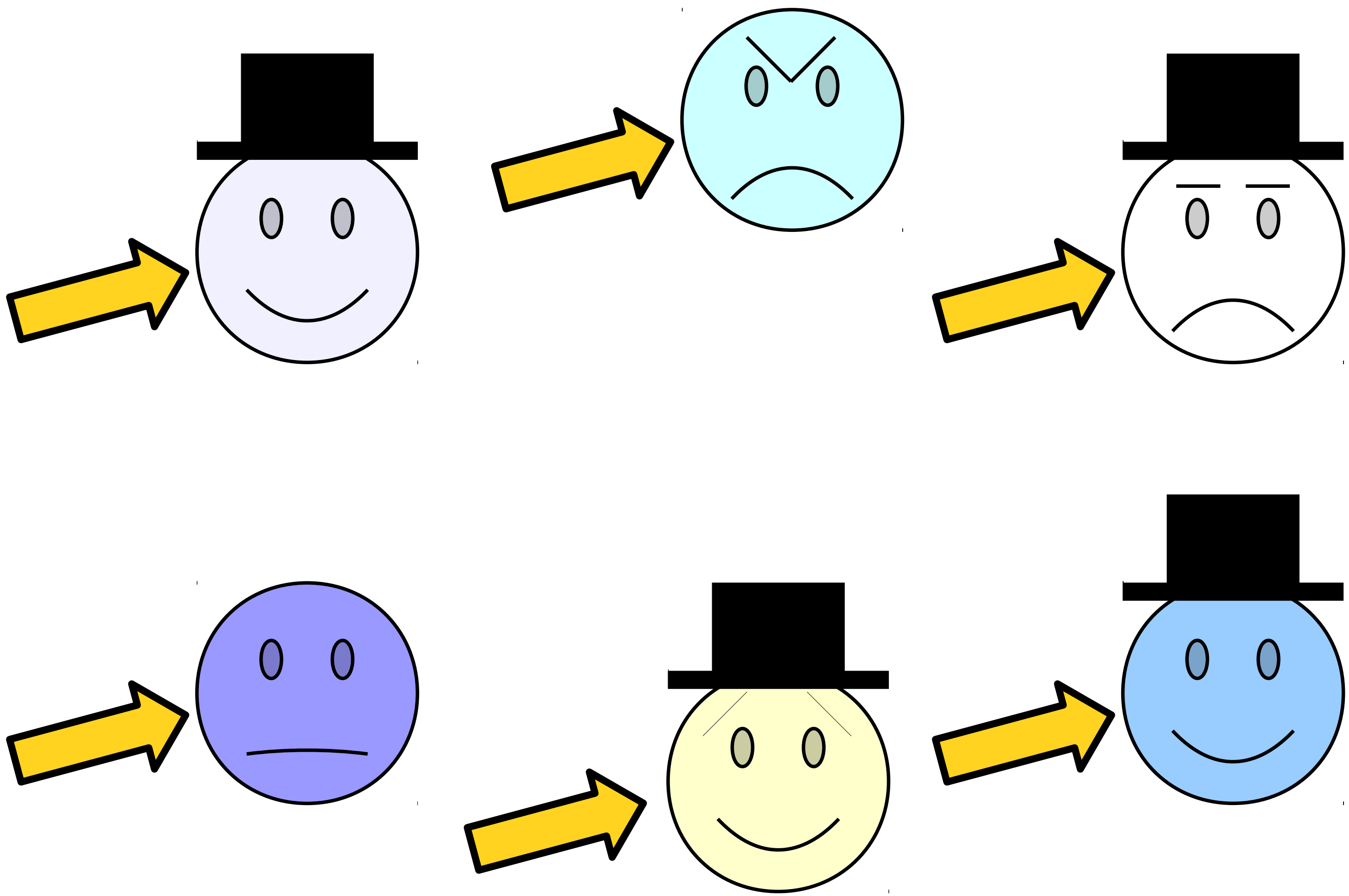

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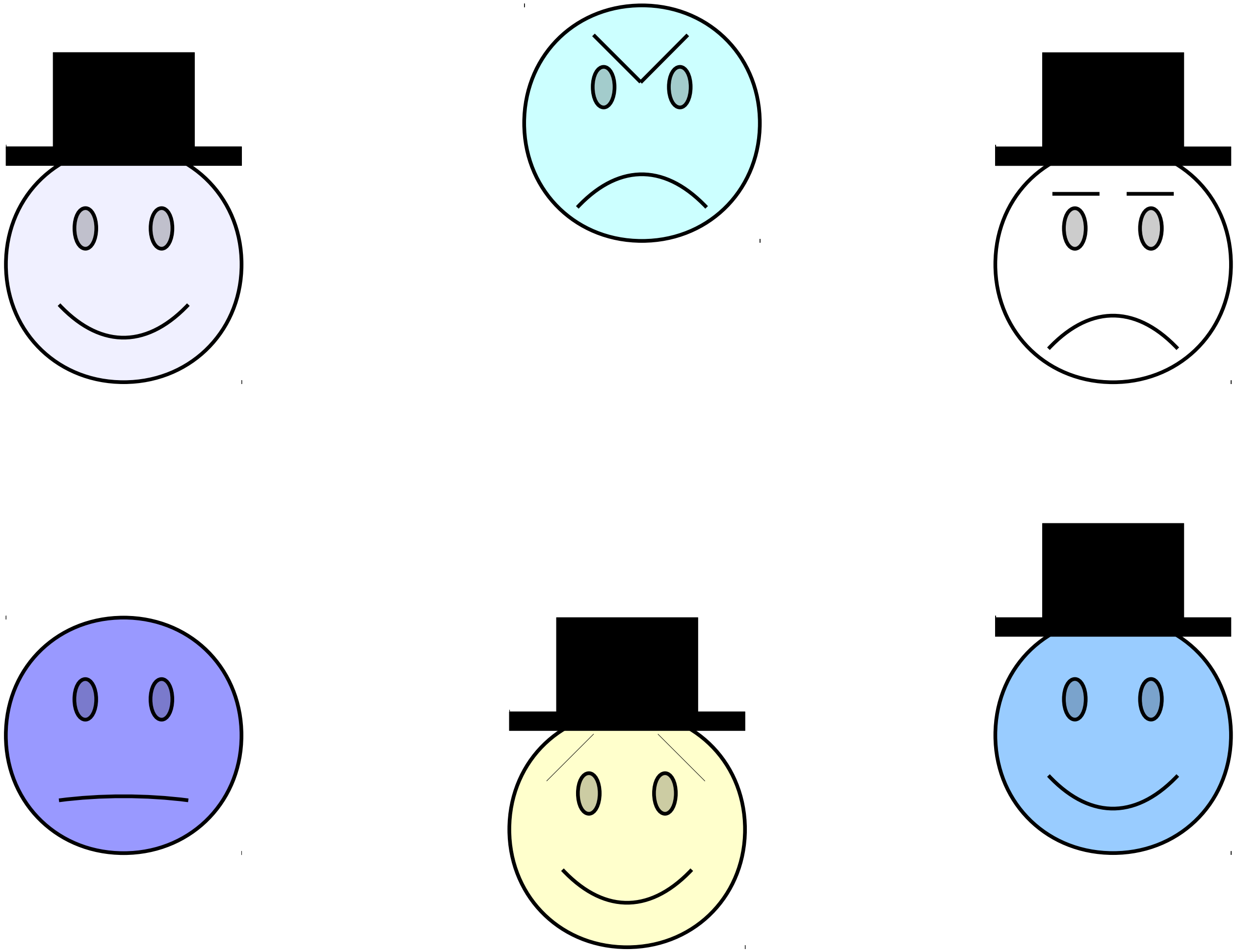

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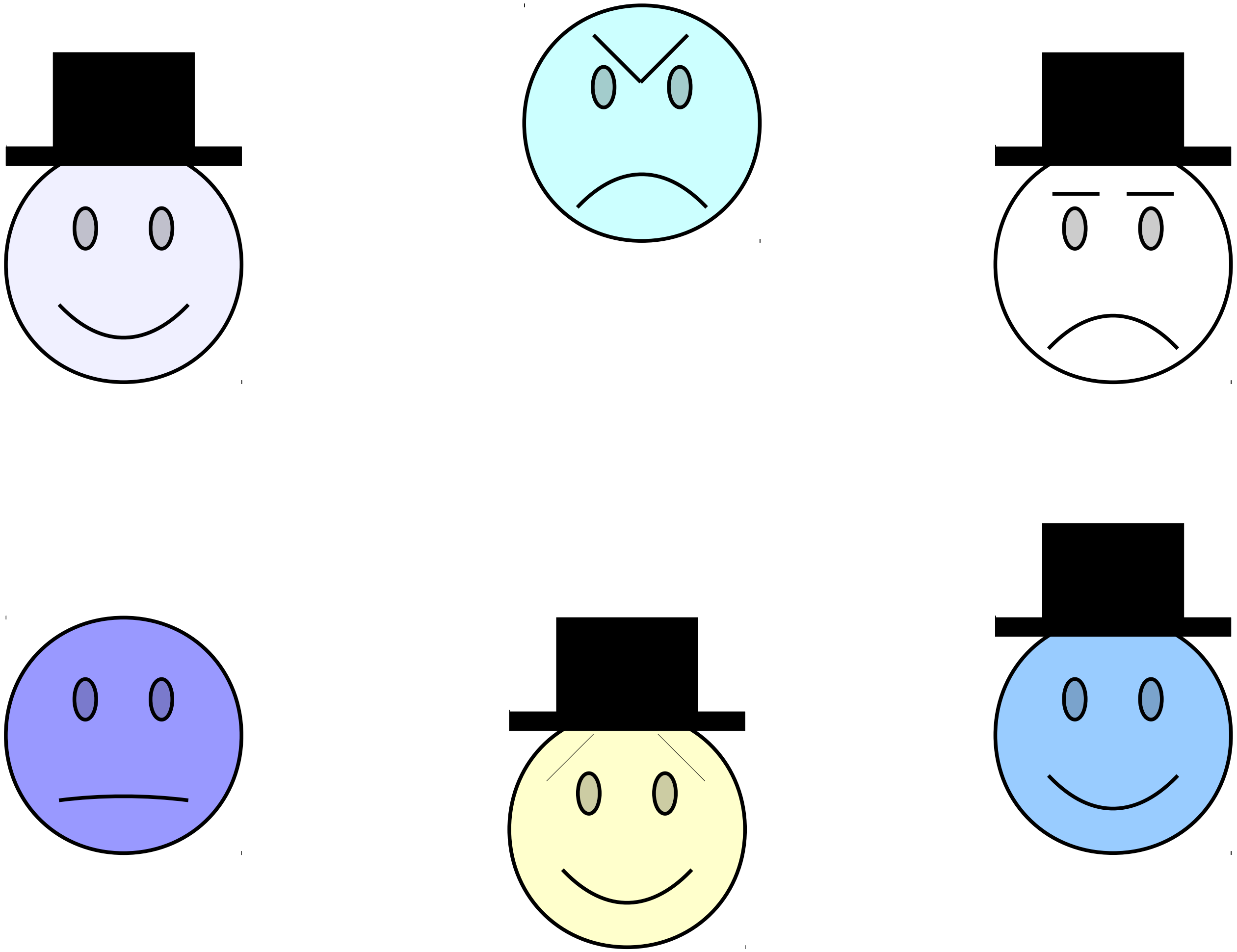

“Every smiling person wears a hat.”	***True***
~~$\forall x.\ (Smiling(x) \land WearingHat(x))$~~	***False***
$\forall x.\ (Smiling(x) \to WearingHat(x))$	***True***

“All *P*'s are *Q*'s”

translates as

$$\forall x.\ (P(x) \to Q(x))$$

Useful Intuition:

Universally-quantified statements are true unless there's a counterexample.

$$\forall x.\ (P(x) \to Q(x))$$

If *x* is a counterexample, it *must* have property *P* but not have property *Q*.

Good Pairings

- The ∀ quantifier *usually* is paired with →.

$$\forall x.\ (P(x) \to Q(x))$$

- The ∃ quantifier *usually* is paired with ∧.

$$\exists x.\ (P(x) \land Q(x))$$

- In the case of ∀, the → connective prevents the statement from being *false* when speaking about some object you don't care about.
- In the case of ∃, the ∧ connective prevents the statement from being *true* when speaking about some object you don't care about.

Proofwriting Workshop

An Incorrect Set Theory Proof

Claim: If A, B, and C are sets and $C \subseteq A \cup B$, then $C \subseteq A$ or $C \subseteq B$.

⚠ ***Incorrect!*** ⚠ ***Proof:*** Consider arbitrary sets A, B, and C where $C \subseteq A \cup B$.

This means that every element of C is in either A or B. If all elements of C are in A, then $C \subseteq A$. Alternately, if everything in C is in B, then $C \subseteq B$. In either case, everything inside of C has to be contained in at least one of these sets, so the theorem is true. ■

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This is just repeating definitions and not making specific claims about specific variables.

Claim: If A, B, and C are sets and $C \subseteq A \cup B$, then $C \subseteq A$ or $C \subseteq B$.

⚠ ***Incorrect!*** ⚠ ***Proof:*** Consider arbitrary sets A, B, and C where $C \subseteq A \cup B$.

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Why is this bad?

Claim: If A, B, and C are sets and $C \subseteq A \cup B$, then $C \subseteq A$ or $C \subseteq B$.

⚠ ***Incorrect!*** ⚠ ***Proof:*** Consider arbitrary sets A, B, and C where $C \subseteq A \cup B$.

This means that every element of C is in either A or B. If all elements of C are in A, then $C \subseteq A$. Alternately, if everything in C is in B, then $C \subseteq B$. In either case, everything inside of C has to be contained in at least one of these sets, so the theorem is true. ■

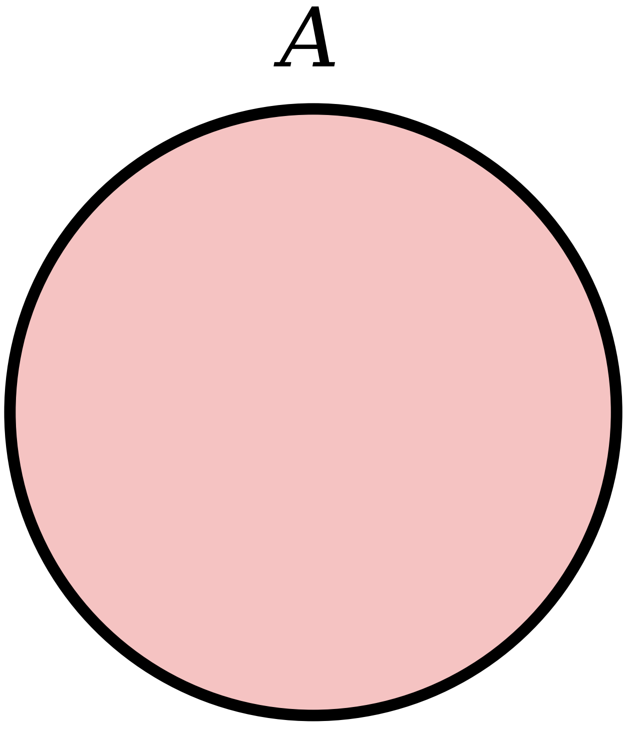
While this claim is true, it does not imply the theorem is true. In fact, this theorem is actually false.

Let's Draw Some Pictures!

Claim: If A, B, and C are sets and $C \subseteq A \cup B$, then $C \subseteq A$ or $C \subseteq B$.

Let's Draw Some Pictures!

Claim: If A, B, and C are sets and $C \subseteq A \cup B$, then $C \subseteq A$ or $C \subseteq B$.



Let's Draw Some Pictures!

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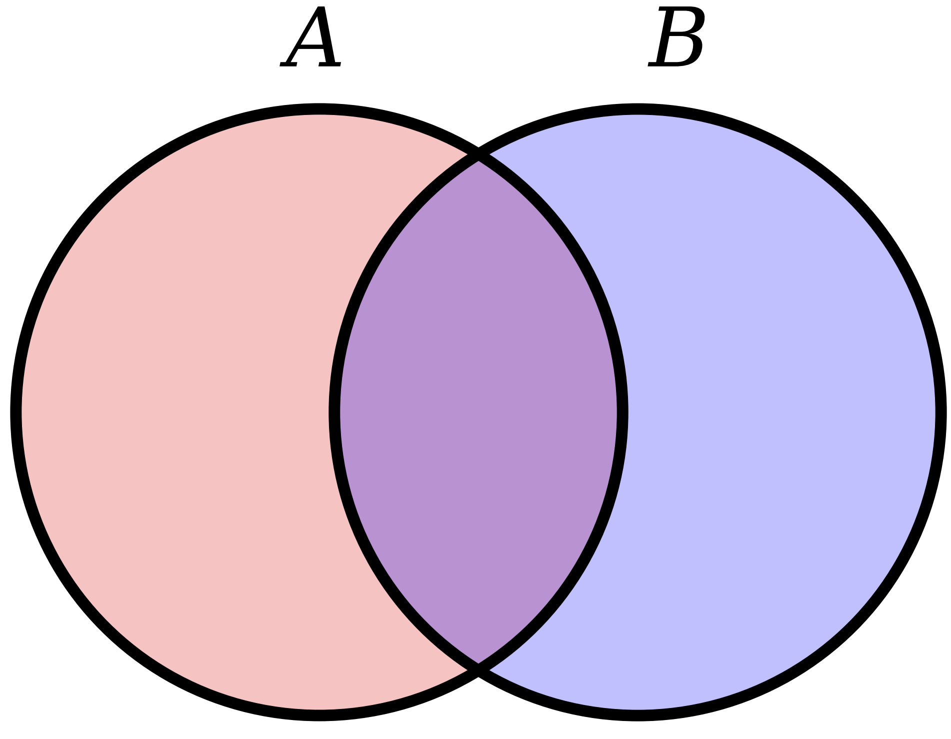
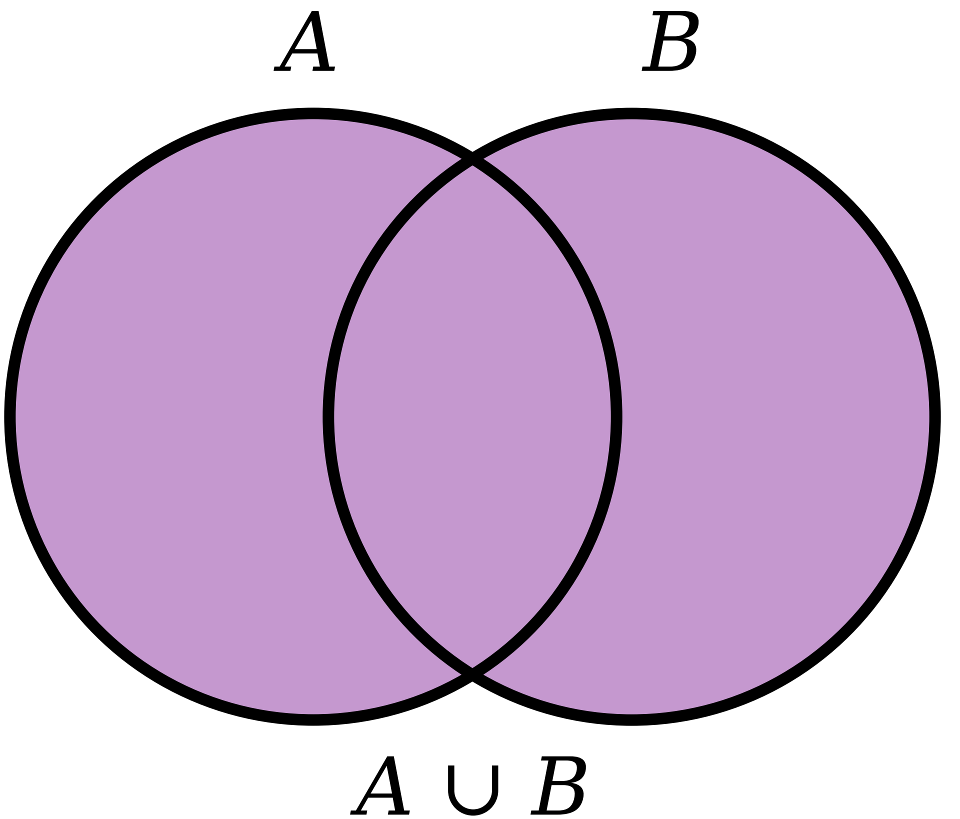


Let's Draw Some Pictures!

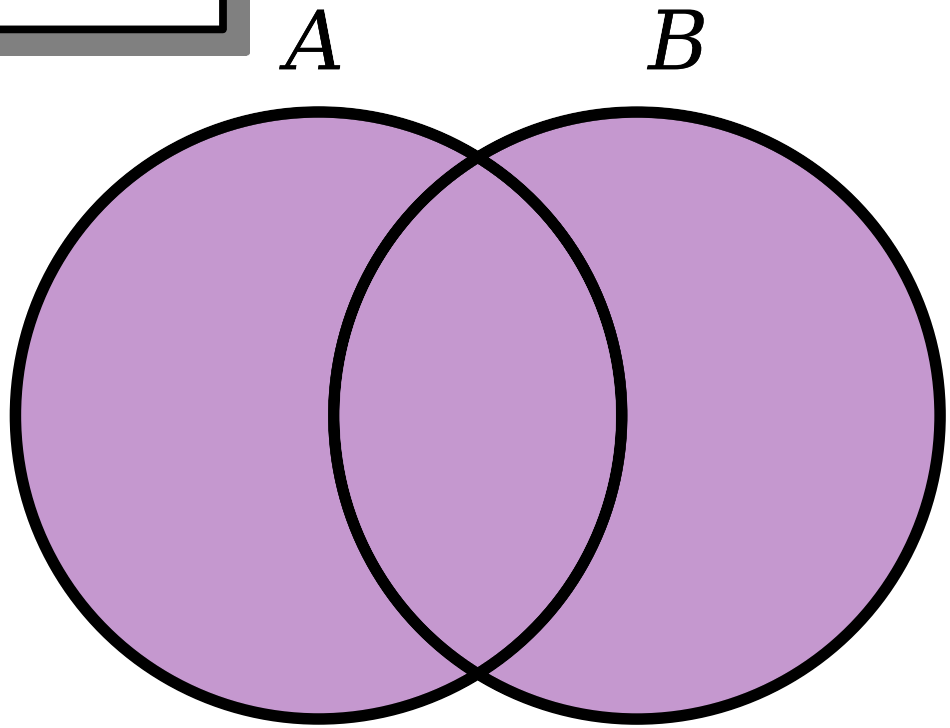
Claim: If A, B, and C are sets and $C \subseteq A \cup B$, then $C \subseteq A$ or $C \subseteq B$.



Let's Draw Some Pictures!

Claim: If A, B, and C are sets and $C \subseteq A \cup B$, then $C \subseteq A$ or $C \subseteq B$.

Recall the intuition of a subset being "something I can circle"



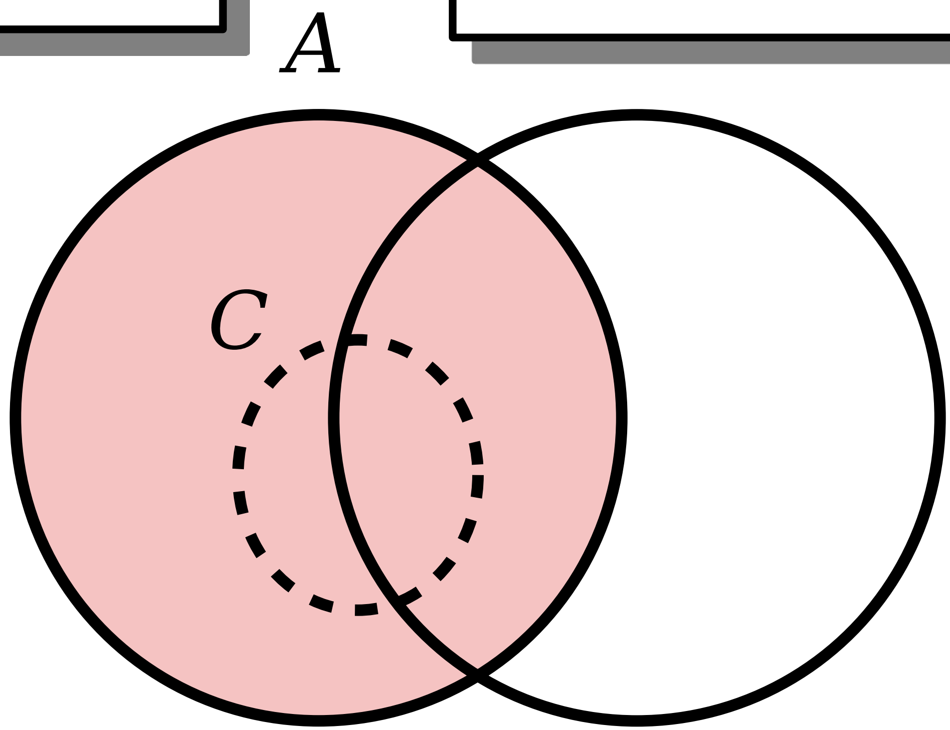
$A \cup B$

Let's Draw Some Pictures!

Claim: If A, B, and C are sets and $C \subseteq A \cup B$, then $C \subseteq A$ or $C \subseteq B$.

Recall the intuition of a subset being "something I can circle"

So C $\subseteq$ A would mean that C is something I can circle in this region.

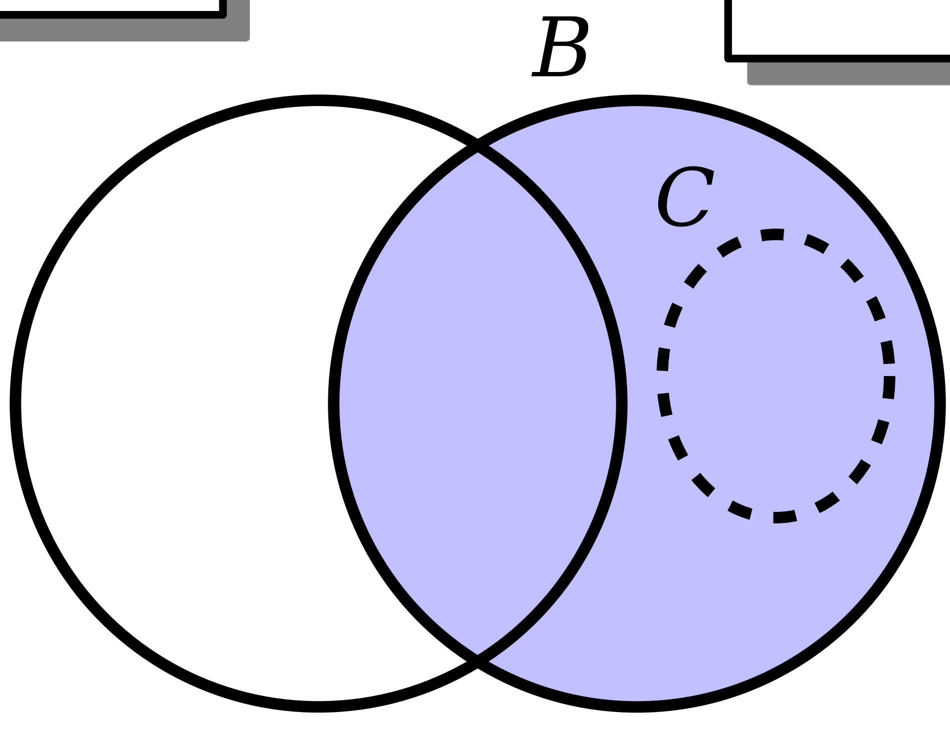


Let's Draw Some Pictures!

Claim: If A, B, and C are sets and $C \subseteq A \cup B$, then $C \subseteq A$ or $C \subseteq B$.

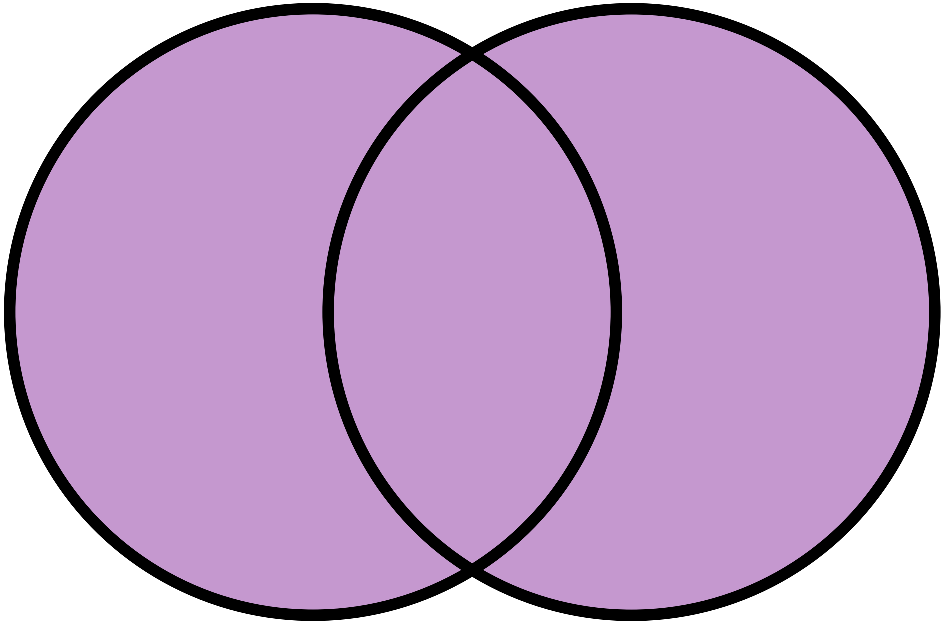
Recall the intuition of a subset being "something I can circle"

Likewise, C $\subseteq$ B would mean that C is something I can circle in this region.



Let's Draw Some Pictures!

Claim: If A, B, and C are sets and $C \subseteq A \cup B$, then $C \subseteq A$ or $C \subseteq B$.

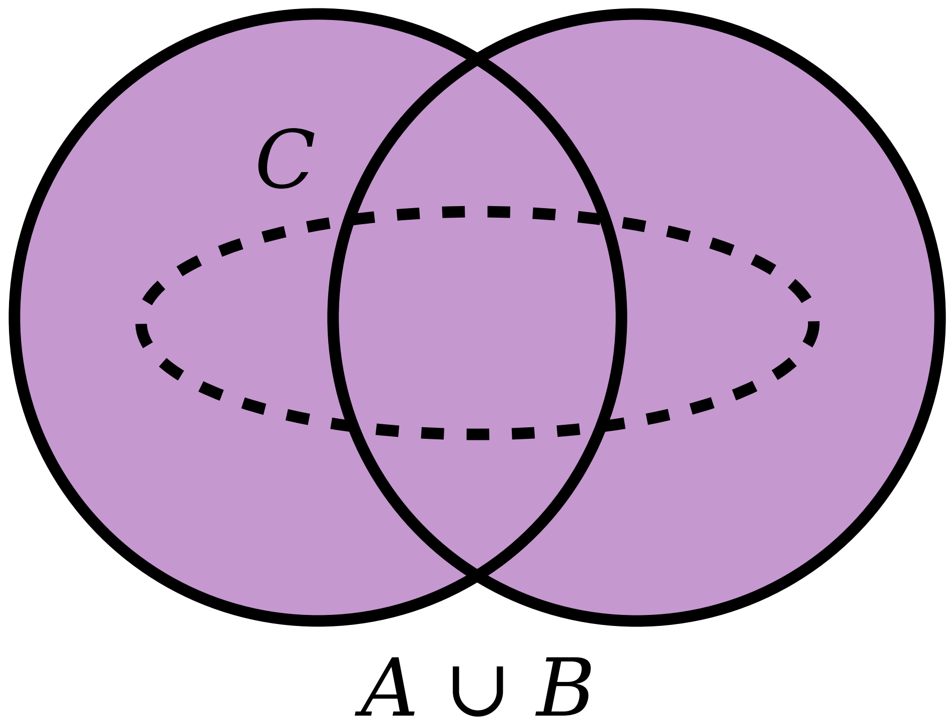


$A \cup B$

Let's Draw Some Pictures!

Claim: If A, B, and C are sets and $C \subseteq A \cup B$, then $C \subseteq A$ or $C \subseteq B$.

But when I look at A∪*B*, I can draw C as a circle containing elements from both A and B!

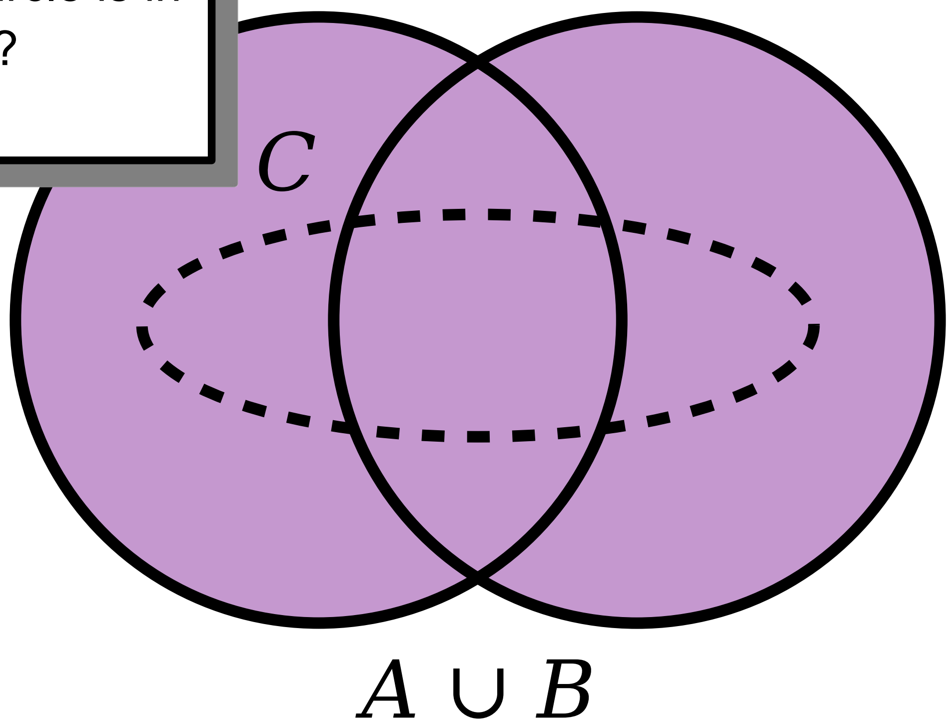


Let’s Draw Some Pictures!

Claim: If A, B, and C are sets and $C \subseteq A \cup B$, then $C \subseteq A$ or $C \subseteq B$.

But when I look at A∪*B*, I can draw C as a circle containing elements from both A and B!

Do you see why this circle is in neither A nor B?

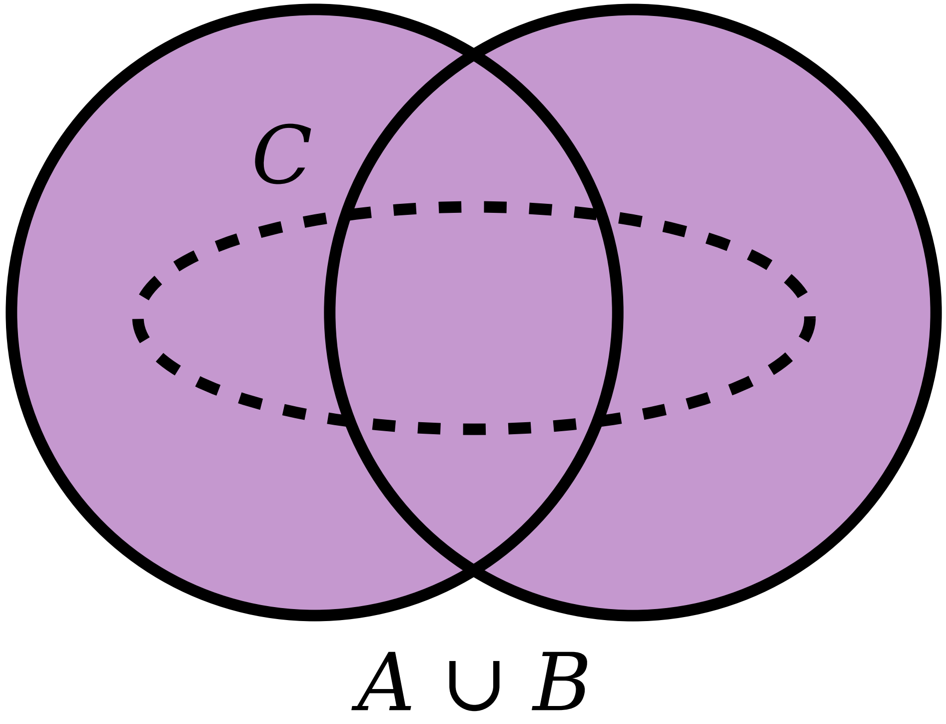


Let's Draw Some Pictures!

Claim: If A, B, and C are sets and $C \subseteq A \cup B$, then $C \subseteq A$ or $C \subseteq B$.

Using this visual intuition, come up with a choice of sets A, B, and C that show this claim is false.

Respond at pollev.com/zhenglian740



Proofs vs. Disproofs

- A ***proof*** is an argument that explains why some ***theorem*** is true.
- A ***disproof*** is an argument that explains why some ***claim*** is false.
- You've seen lots of examples of proofs. What does a disproof look like?

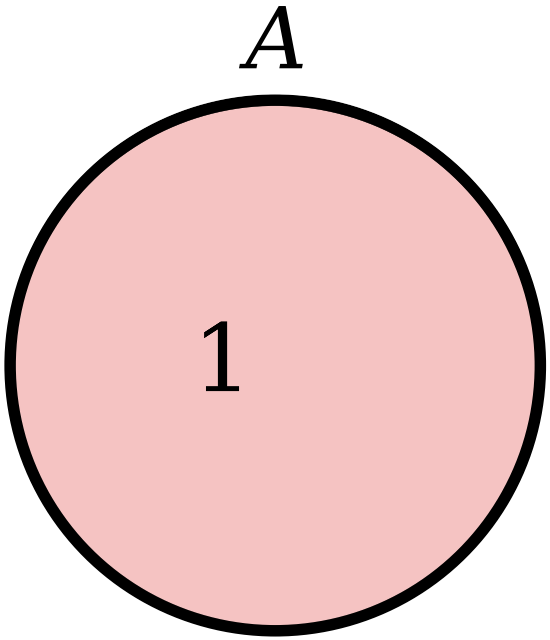
Claim: If A, B, and C are sets and $C \subseteq A \cup B$, then $C \subseteq A$ or $C \subseteq B$.

Claim: If A, B, and C are sets and $C \subseteq A \cup B$, then $C \subseteq A$ or $C \subseteq B$.

Disproof: We will show that there are sets A, B, and C where $C \subseteq A \cup B$, but $C \nsubseteq A$ and $C \nsubseteq B$.

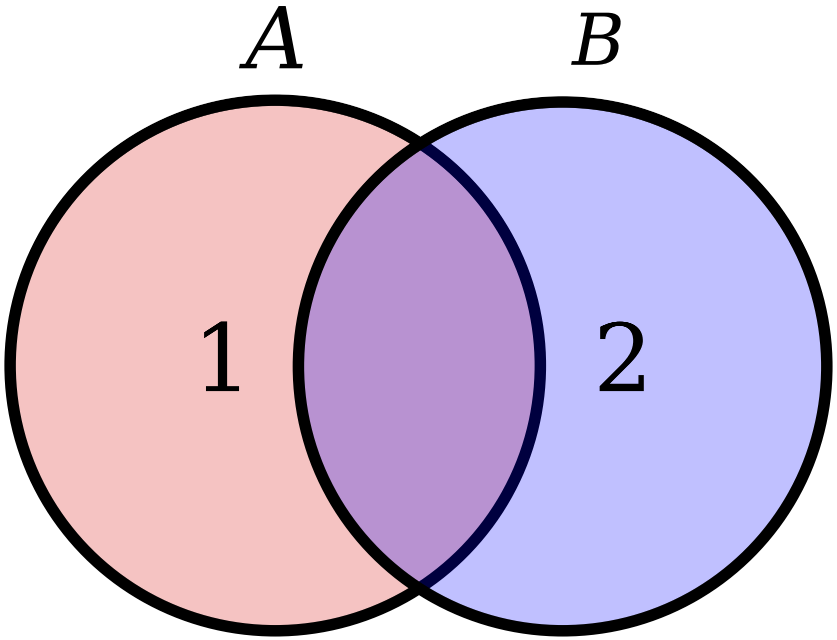
Claim: If A, B, and C are sets and $C \subseteq A \cup B$, then $C \subseteq A$ or $C \subseteq B$.

Disproof: We will show that there are sets A, B, and C where $C \subseteq A \cup B$, but $C \nsubseteq A$ and $C \nsubseteq B$. Consider the sets $A = \{1\}$



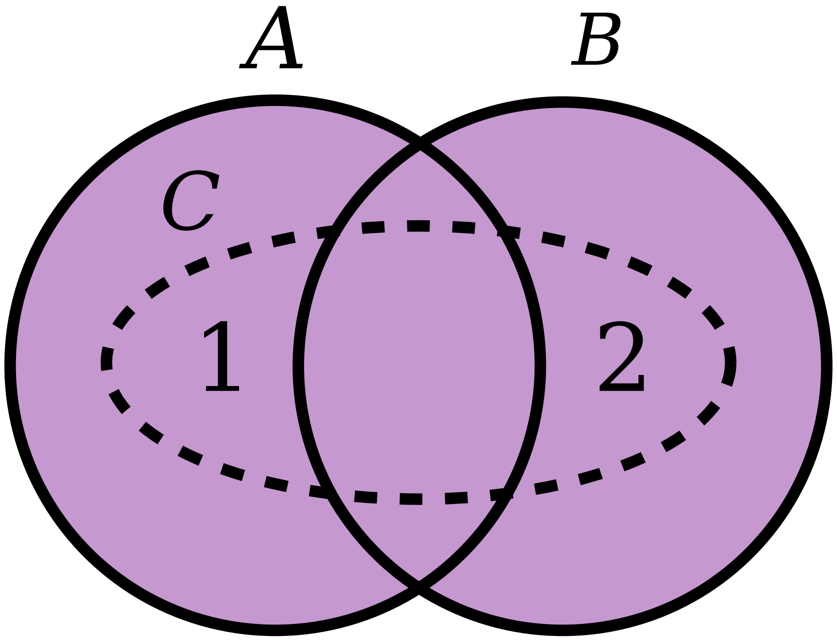
Claim: If A, B, and C are sets and $C \subseteq A \cup B$, then $C \subseteq A$ or $C \subseteq B$.

Disproof: We will show that there are sets A, B, and C where $C \subseteq A \cup B$, but $C \nsubseteq A$ and $C \nsubseteq B$. Consider the sets $A = \{1\}$, $B = \{2\}$



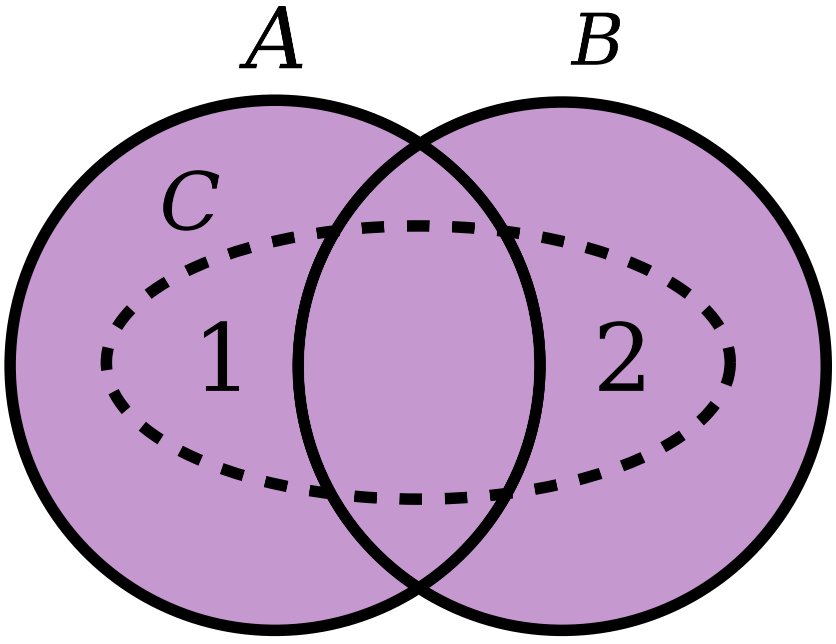
Claim: If A, B, and C are sets and $C \subseteq A \cup B$, then $C \subseteq A$ or $C \subseteq B$.

Disproof: We will show that there are sets A, B, and C where $C \subseteq A \cup B$, but $C \not\subseteq A$ and $C \not\subseteq B$. Consider the sets $A = \{1\}$, $B = \{2\}$, and $C = \{1, 2\}$.



Claim: If A, B, and C are sets and $C \subseteq A \cup B$, then $C \subseteq A$ or $C \subseteq B$.

Disproof: We will show that there are sets A, B, and C where $C \subseteq A \cup B$, but $C \not\subseteq A$ and $C \not\subseteq B$. Consider the sets $A = \{1\}$, $B = \{2\}$, and $C = \{1, 2\}$. Now notice that $\{1, 2\} \subseteq A \cup B$ so $C \subseteq A \cup B$



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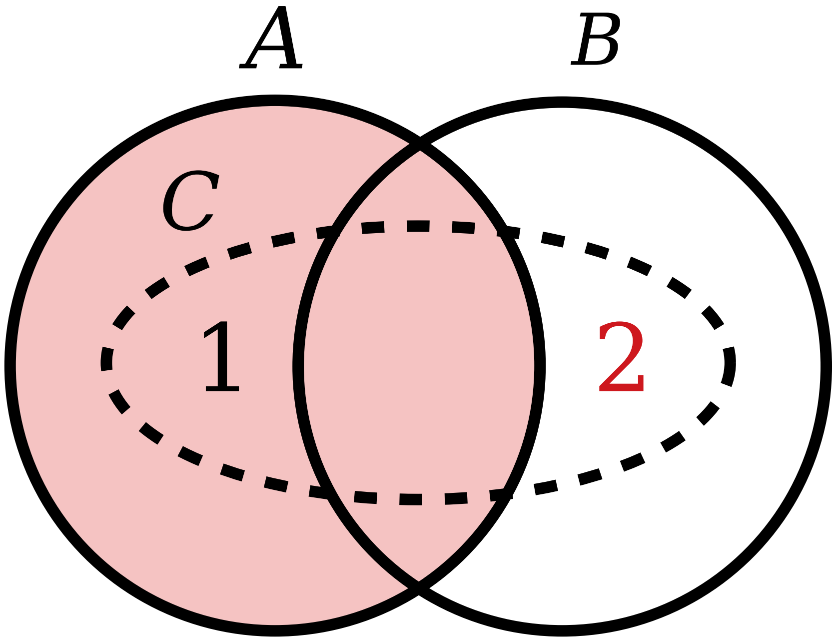
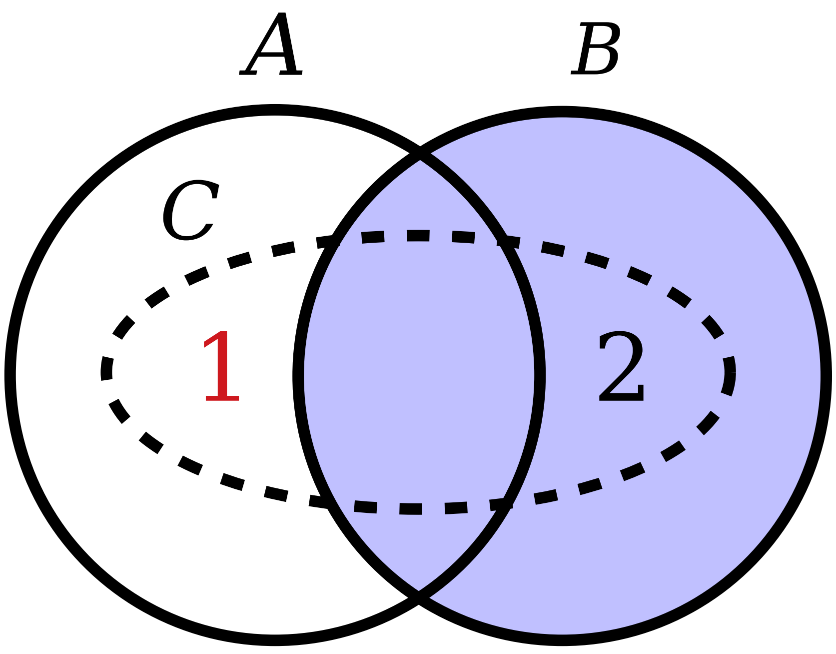
Disproof: We will show that there are sets A, B, and C where $C \subseteq A \cup B$, but $C \nsubseteq A$ and $C \nsubseteq B$. Consider the sets $A = \{1\}$, $B = \{2\}$, and $C = \{1, 2\}$. Now notice that $\{1, 2\} \subseteq A \cup B$ so $C \subseteq A \cup B$, but $C \nsubseteq A$ because $2 \in C$ but $2 \notin A$



Claim: If A, B, and C are sets and $C \subseteq A \cup B$, then $C \subseteq A$ or $C \subseteq B$.

Disproof: We will show that there are sets A, B, and C where $C \subseteq A \cup B$, but $C \not\subseteq A$ and $C \not\subseteq B$. Consider the sets $A = \{1\}$, $B = \{2\}$, and $C = \{1, 2\}$. Now notice that $\{1, 2\} \subseteq A \cup B$ so $C \subseteq A \cup B$, but $C \not\subseteq A$ because $2 \in C$ but $2 \notin A$, and $C \not\subseteq B$ because $1 \in C$ but $1 \notin B$.



Claim: If A, B, and C are sets and $C \subseteq A \cup B$, then $C \subseteq A$ or $C \subseteq B$.

Disproof: We will show that there are sets A, B, and C where $C \subseteq A \cup B$, but $C \not\subseteq A$ and $C \not\subseteq B$. Consider the sets $A = \{1\}$, $B = \{2\}$, and $C = \{1, 2\}$. Now notice that $\{1, 2\} \subseteq A \cup B$ so $C \subseteq A \cup B$, but $C \not\subseteq A$ because $2 \in C$ but $2 \notin A$, and $C \not\subseteq B$ because $1 \in C$ but $1 \notin B$.

Thus we've found a set C which is a subset of $A \cup B$ but is not a subset of either A or B, which is what we needed to show. ■

Proofwriting Advice

- Be ***very wary*** of proofs that speak generally about “all objects” of a particular type.
  - As you’ve just seen, it’s easy to accidentally prove a false statement at this level of detail.
  - Making broad, high-level claims often indicates deeper logic errors or conceptual misunderstanding (like *code smell* but for proofs!)

Proofwriting Advice

A Very Good Idea: After you've written a draft of a proof, run through all of the points on the Proofwriting Checklist.

- This is a *great* exercise that you can do with a partner!

Proofs on Subsets

Theorem: If A, B, and C are sets and $C \subseteq A \cap B$, then $C \subseteq A$ and $C \subseteq B$.

Theorem: If A, B, and C are sets and $C \subseteq A \cap B$, then $C \subseteq A$ and $C \subseteq B$.

Hold on, isn't this the claim we just disproved?

Theorem: If A, B, and C are sets and $C \subseteq A \cap B$, then $C \subseteq A$ and $C \subseteq B$.

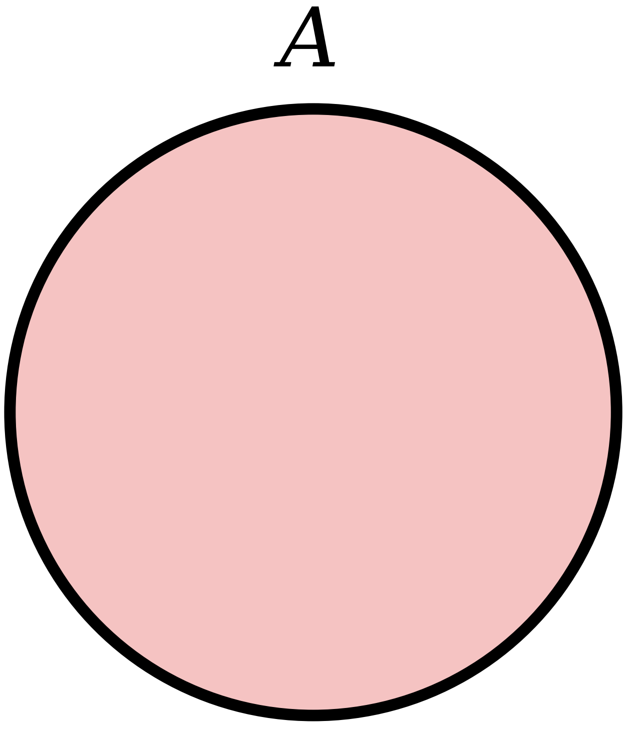
Notice that that's an intersection, not a union! It turns out that this claim is actually true.

Let's Draw Some Pictures!

Theorem: If A, B, and C are sets and $C \subseteq A \cap B$, then $C \subseteq A$ and $C \subseteq B$.

Let's Draw Some Pictures!

Theorem: If A, B, and C are sets and $C \subseteq A \cap B$, then $C \subseteq A$ and $C \subseteq B$.



Let's Draw Some Pictures!

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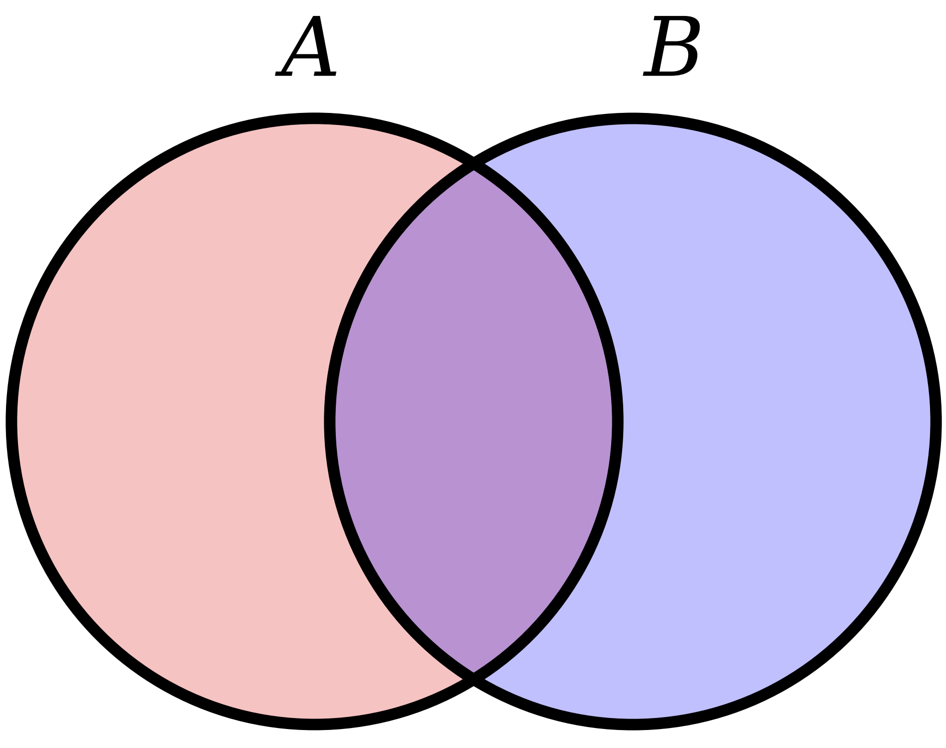
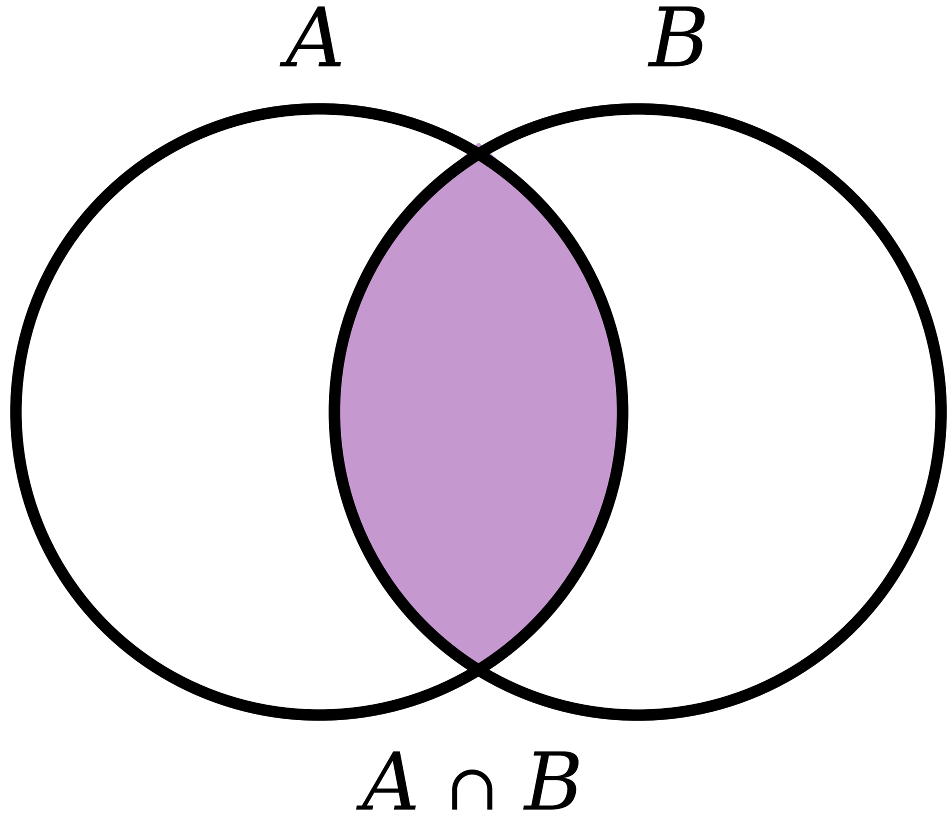


Let's Draw Some Pictures!

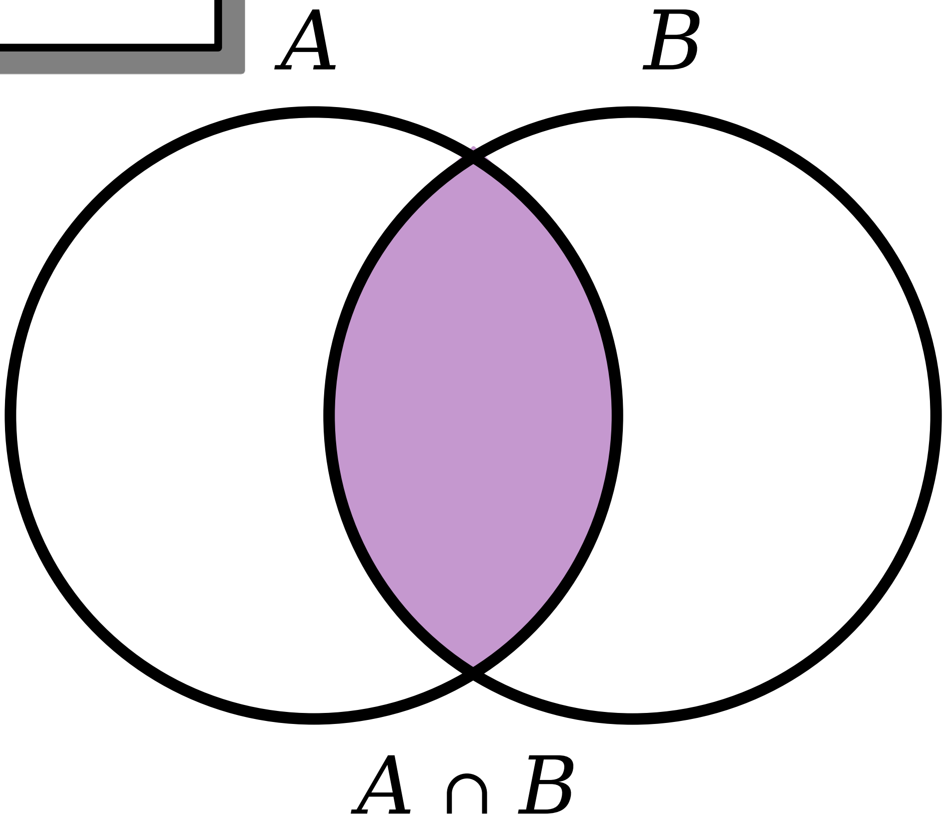
Theorem: If A, B, and C are sets and $C \subseteq A \cap B$, then $C \subseteq A$ and $C \subseteq B$.



Let's Draw Some Pictures!

Theorem: If A, B, and C are sets and $C \subseteq A \cap B$, then $C \subseteq A$ and $C \subseteq B$.

Recall the intuition of a subset being "something I can circle"

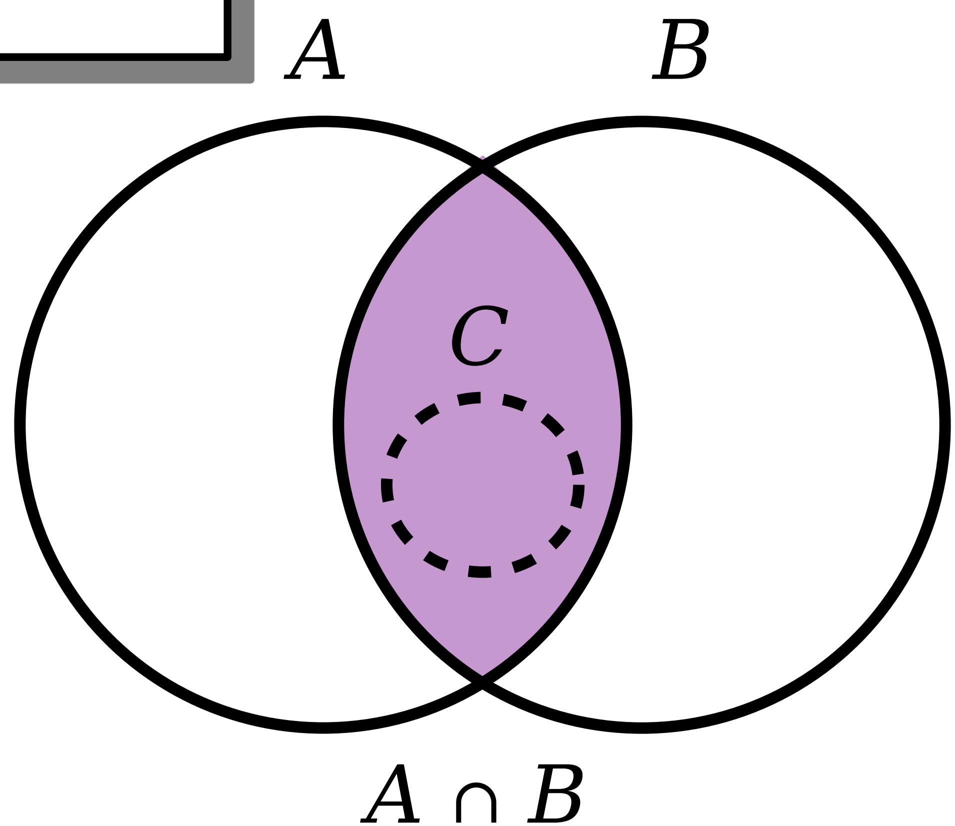


Let's Draw Some Pictures!

Theorem: If A, B, and C are sets and $C \subseteq A \cap B$, then $C \subseteq A$ and $C \subseteq B$.

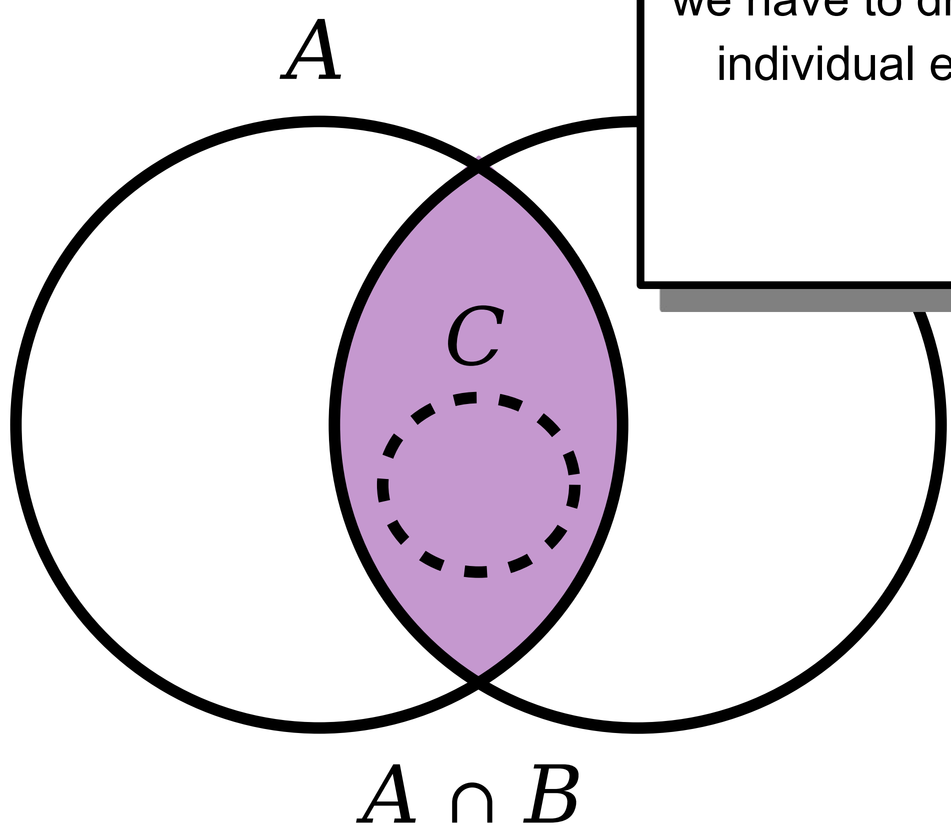
Recall the intuition of a subset being "something I can circle"

When I look at A⋂*B*, any circle I can draw in this region can be found in both A and B.



Let's Draw Some Pictures!

Theorem: If A, B, and C are sets and $C \subseteq A \cap B$, then $C \subseteq A$ and $C \subseteq B$.



This is a great visual intuition to see why the theorem is true. Now we have to drill down to the level of individual elements to write the proof.

Theorem: If A, B, and C are sets and $C \subseteq A \cap B$, then $C \subseteq A$ and $C \subseteq B$.

What We're Assuming	*What We Need To Show*

When confronted with a theorem to prove, the first step is to make sure you understand where you're
starting and where you're going.

Theorem: If A, B, and C are sets and $C \subseteq A \cap B$, then $C \subseteq A$ and $C \subseteq B$.

What We're Assuming

- A, B, and C are sets
- $C \subseteq A \cap B$

What We Need To Show

- $C \subseteq A$ and $C \subseteq B$

Theorem: If A, B, and C are sets and $C \subseteq A \cap B$, then $C \subseteq A$ and $C \subseteq B$.

What We're Assuming

- A, B, and C are sets
- $C \subseteq A \cap B$

What We Need To Show

- $C \subseteq A$ and $C \subseteq B$

A great proofwriting strategy is to ***write down relevant definitions***. This gives you a better sense of what you need to prove and what tools you have at hand.

Theorem: If A, B, and C are sets and $C \subseteq A \cap B$, then $C \subseteq A$ and $C \subseteq B$.

What We're Assuming

- A, B, and C are sets
- C ⊆ A∩B

What We Need To Show

- C ⊆ A and C ⊆ B

Before we start:
- What is the definition of subset?
- How do you prove that one set is a subset of another?
- If you know that one set is a subset of another, what can you conclude?

Theorem: If A, B, and C are sets and $C \subseteq A \cap B$, then $C \subseteq A$ and $C \subseteq B$.

What We're Assuming

- A, B, and C are sets
- $C \subseteq A \cap B$

What We Need To Show

- $C \subseteq A$ and $C \subseteq B$

Definition: If S and T are sets, then $S \subseteq T$ when for every $x \in S$, we have $x \in T$.

Theorem: If A, B, and C are sets and $C \subseteq A \cap B$, then $C \subseteq A$ and $C \subseteq B$.

What We're Assuming

- A, B, and C are sets
- $C \subseteq A \cap B$

What We Need To Show

- $C \subseteq A$ and $C \subseteq B$

Definition: If S and T are sets, then $S \subseteq T$ when for every $x \in S$, we have $x \in T$.

To prove that $S \subseteq T$:

Pick an arbitrary $x \in S$, then prove $x \in T$.

If you know that $S \subseteq T$:

If you have an $x \in S$, you can conclude $x \in T$.

Theorem: If A, B, and C are sets and $C \subseteq A \cap B$, then $C \subseteq A$ and $C \subseteq B$.

What We're Assuming

- A, B, and C are sets
- C ⊆ A∩B

What We Need To Show

- C ⊆ A and C ⊆ B

Definition: If S and T are sets, then $S \subseteq T$ when for every $x \in S$, we have $x \in T$.

To prove that $S \subseteq T$:

Pick an arbitrary $x \in S$, then prove $x \in T$.

If you know that $S \subseteq T$:

If you have an $x \in S$, you can conclude $x \in T$.

Theorem: If A, B, and C are sets and $C \subseteq A \cap B$, then $C \subseteq A$ and $C \subseteq B$.

This reading of the definition is usually helpful for unpacking this column!

What We Need To Show

- $C \subseteq A$ and $C \subseteq B$

Definition: If S and T are sets, then $S \subseteq T$ when for every $x \in S$, we have $x \in T$.

To prove that $S \subseteq T$:

Pick an arbitrary $x \in S$, then prove $x \in T$.

If you know that $S \subseteq T$:

If you have an $x \in S$, you can conclude $x \in T$.

Theorem: If A, B, and C are sets and $C \subseteq A \cap B$, then $C \subseteq A$ and $C \subseteq B$.

What We're Assuming

- A, B, and C are sets
- $C \subseteq A \cap B$

What We Need To Show

- $C \subseteq A$ and $C \subseteq B$

Our Tools

- In general to show that $S \subseteq T$, pick an arbitrary $x \in S$, show that $x \in T$

Theorem: If A, B, and C are sets and $C \subseteq A \cap B$, then $C \subseteq A$ and $C \subseteq B$.

What We're Assuming

- A, B, and C are sets
- $C \subseteq A \cap B$

What We Need To Show

- $C \subseteq A$ and $C \subseteq B$

Our Tools

- In general to show that $S \subseteq T$, pick an arbitrary $x \in S$, show that $x \in T$

How can we apply this general template to our specific problem?

Theorem: If A, B, and C are sets and $C \subseteq A \cap B$, then $C \subseteq A$ and $C \subseteq B$.

What We're Assuming

- A, B, and C are sets
- $C \subseteq A \cap B$

Our Tools

- In general to show that $S \subseteq T$, pick an arbitrary $x \in S$, show that $x \in T$

What We Need To Show

- $C \subseteq A$ and $C \subseteq B$
  - Pick an $x \in C$, show that $x \in A$
  - Pick an $x \in C$, show that $x \in B$

Theorem: If A, B, and C are sets and $C \subseteq A \cap B$, then $C \subseteq A$ and $C \subseteq B$.

What We're Assuming

- A, B, and C are sets
- $C \subseteq A \cap B$

Our Tools

- In general to show that $S \subseteq T$, pick an arbitrary $x \in S$, show that $x \in T$

What We Need To Show

- $C \subseteq A$ and $C \subseteq B$
  - Pick an $x \in C$, show that $x \in A$
  - Pick an $x \in C$, show that $x \in B$

Now we know that ultimately, we're going to have to do these two things. Let's see what tools we have that can get us here!

Theorem: If A, B, and C are sets and $C \subseteq A \cap B$, then $C \subseteq A$ and $C \subseteq B$.

What We're Assuming

- A, B, and C are sets
- $C \subseteq A \cap B$

Our Tools

- In general to show that $S \subseteq T$, pick an arbitrary $x \in S$, show that $x \in T$

What We Need To Show

- $C \subseteq A$ and $C \subseteq B$
  - Pick an $x \in C$, show that $x \in A$
  - Pick an $x \in C$, show that $x \in B$

Theorem: If A, B, and C are sets and $C \subseteq A \cap B$, then $C \subseteq A$ and $C \subseteq B$.

What We're Assuming

- A, B, and C are sets
- $C \subseteq A \cap B$

This reading of the definition is usually helpful for unpacking this column!

at x

∈ A

that x

Definition: If S and T are sets, then $S \subseteq T$ when for every $x \in S$, we have $x \in T$.

To prove that S ⊆ T:
Pick an arbitrary $x \in S$, then prove $x \in T$.

If you know that S ⊆ T:
If you have an $x \in S$, you can conclude $x \in T$.

Theorem: If A, B, and C are sets and $C \subseteq A \cap B$, then $C \subseteq A$ and $C \subseteq B$.

What We're Assuming

- A, B, and C are sets
- $C \subseteq A \cap B$

Our Tools

- In general to show that $S \subseteq T$, pick an arbitrary $x \in S$, show that $x \in T$

What We Need To Show

- $C \subseteq A$ and $C \subseteq B$
    - Pick an $x \in C$, show that $x \in A$
    - Pick an $x \in C$, show that $x \in B$

Before we continue:
- What is the definition of set intersection?

Theorem: If A, B, and C are sets and $C \subseteq A \cap B$, then $C \subseteq A$ and $C \subseteq B$.

What We're Assuming

- A, B, and C are sets
- $C \subseteq A \cap B$

What We Need To Show

- $C \subseteq A$ and $C \subseteq B$
  - Pick an $x \in C$, show that $x \in A$

Definition: The set $S \cap T$ is the set where, for any x:

$$x \in S \cap T \quad \text{when} \quad x \in S \text{ and } x \in T$$

Theorem: If A, B, and C are sets and $C \subseteq A \cap B$, then $C \subseteq A$ and $C \subseteq B$.

What We're Assuming	*What We Need To Show*
• A, B, and C are sets • $C \subseteq A \cap B$	• $C \subseteq A$ and $C \subseteq B$ • Pick an $x \in C$, show that $x \in A$

Definition: The set $S \cap T$ is the set where, for any x:

$x \in S \cap T$ when $x \in S$ and $x \in T$

To prove that x ∈ S ∩ T:

Prove both that $x \in S$ and that $x \in T$.

If you know that x ∈ S ∩ T:

You can conclude both that $x \in S$ and that $x \in T$.

Theorem: If A, B, and C are sets and $C \subseteq A \cap B$, then $C \subseteq A$ and $C \subseteq B$.

What We're Assuming

- A, B, and C are sets
- $C \subseteq A \cap B$

This is the one we want!

- Pick an $x \in C$, show that $x \in A$
- Pick an $x \in C$, show that x

Definition: The set $S \cap T$ is the set where, for any x:

$$x \in S \cap T \quad \text{when} \quad x \in S \text{ and } x \in T$$

To prove that $x \in S \cap T$:

Prove both that $x \in S$ and that $x \in T$.

If you know that $x \in S \cap T$:

You can conclude both that $x \in S$ and that $x \in T$.

Theorem: If A, B, and C are sets and $C \subseteq A \cap B$, then $C \subseteq A$ and $C \subseteq B$.

What We're Assuming

- A, B, and C are sets
- $C \subseteq A \cap B$

Our Tools

- In general to show that $S \subseteq T$, pick an arbitrary $x \in S$, show that $x \in T$.
- If you know that $S \subseteq T$ and you have an $x \in S$, you can conclude $x \in T$.
- If you know that $x \in S \cap T$, we can conclude that $x \in S$ and $x \in T$.

What We Need To Show

- $C \subseteq A$ and $C \subseteq B$
  - Pick an $x \in C$, show that $x \in A$
  - Pick an $x \in C$, show that $x \in B$

Theorem: If A, B, and C are sets and $C \subseteq A \cap B$, then $C \subseteq A$ and $C \subseteq B$.

What We're Assuming

- A, B, and C are sets
- $C \subseteq A \cap B$

Our Tools

- In general to show that $S \subseteq T$, pick an arbitrary $x \in S$, show that $x \in T$.
- If you know that $S \subseteq T$ and you have an $x \in S$, you can conclude $x \in T$.
- If you know that $x \in S \cap T$, we can conclude that $x \in S$ and $x \in T$.

What We Need To Show

- $C \subseteq A$ and $C \subseteq B$
  - Pick an $x \in C$, show that $x \in A$
  - Pick an $x \in C$, show that $x \in B$

Let's go and try and do the proof with what we've got here!

Theorem: If A, B, and C are sets and $C \subseteq A \cap B$, then $C \subseteq A$ and $C \subseteq B$.

Rough Outline

- Assume $C \subseteq A \cap B$

Relevant Definitions

- In general to show that $S \subseteq T$, pick an arbitrary $x \in S$, show that $x \in T$
- If you know that $S \subseteq T$ and you have an $x \in S$, you can conclude $x \in T$.
- If you know that $x \in S \cap T$, we can conclude that $x \in S$ and $x \in T$.

Theorem: If A, B, and C are sets and $C \subseteq A \cap B$, then $C \subseteq A$ and $C \subseteq B$.

Rough Outline

- Assume C ⊆ A ∩ B
- Proving C ⊆ A
  - Pick an x ∈ C
  - Conclude x ∈ A

Relevant Definitions

- In general to show that $S \subseteq T$, pick an arbitrary $x \in S$, show that $x \in T$
- If you know that $S \subseteq T$ and you have an $x \in S$, you can conclude $x \in T$.
- If you know that $x \in S \cap T$, we can conclude that $x \in S$ and $x \in T$.

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  - Pick an x ∈ C
  - What goes here?
  - Conclude x ∈ A

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  - Pick an $x \in C$
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  - x ∈ A ∩ B
  - x ∈ A and x ∈ B
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Theorem: If A, B, and C are sets and $C \subseteq A \cap B$, then $C \subseteq A$ and $C \subseteq B$.

Rough Outline

- Assume $C \subseteq A \cap B$
- Proving $C \subseteq A$
  - Pick an $x \in C$
  - $x \in A \cap B$
  - $x \in A$ and $x \in B$
  - Conclude $x \in A$

We also need to prove that $C \subseteq B$.

Notice that if you take the outline here and literally swap the variable A for the variable B, you get a working proof.

Theorem: If A, B, and C are sets and $C \subseteq A \cap B$, then $C \subseteq A$ and $C \subseteq B$.

Rough Outline

- Assume C ⊆ B ∩ A
- Proving C ⊆ B
  - Pick an x ∈ C
  - x ∈ B ∩ A
  - x ∈ B and x ∈ A
  - Conclude x ∈ B

In a case like this where your proof would have two completely symmetric branches, it's fine to write up just one and say "***by symmetry***, [the other branch] is also true."

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Rough Outline

- Assume C ⊆ A ∩ B
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  - Pick an x ∈ C
  - x ∈ A ∩ B
  - x ∈ A and x ∈ B
  - Conclude x ∈ A

Try it yourself: Take a few minutes and write up a proof of the theorem using this outline.

Then share your proof with your neighbors and critique each other!

Respond at pollev.com/zhenglian740

Theorem: If A, B, and C are sets and $C \subseteq A \cap B$, then $C \subseteq A$ and $C \subseteq B$.

Proof: Let A, B, and C be arbitrary sets where $C \subseteq A \cap B$. We need to show that $C \subseteq A$ and $C \subseteq B$. Because the roles of A and B in this proof are symmetric, we can just prove that $C \subseteq A$.

Choose any element $x \in C$. Since $C \subseteq A \cap B$, we know that $x \in A \cap B$. This tells us that $x \in A$ and $x \in B$. In particular, this means that $x \in A$, thus completing the proof. ■

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Are you clearly stating what you're assuming and what you're trying to prove?

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Are you making specific claims about specific variables? Your proof should NOT have statements of the form “every element of C”.

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Are all variables properly introduced and scoped? You should be able to point at every variable and say that it is either:

1) an arbitrarily chosen value – owned by the reader
2) an existentially instantiated value – owned by no one
3) an explicitly chosen value – owned by you (the proof writer)

Next Time

- ***First-Order Translations***
  - How do we translate from English into first-order logic?
- ***Quantifier Orderings***
  - How do you select the order of quantifiers in first-order logic formulas?
- ***Negating Formulas***
  - How do you mechanically determine the negation of a first-order formula?
- ***Expressing Uniqueness***
  - How do we say there's just one object of a certain type?